HK.NCCP

International Journal of Applied Mathematics and Soft Computing
Volume 6, Issue 2, August, 2020

Notes on Spectral Decomposition and Star Complements of Randić Matrix

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Abstract: In this paper, we mainly resort out some results of spectral decomposition and star complements of Randić matrix of a graph. Additionally, we give a relation between spectral decomposition and star complements of Randić matrix of a graph, and some further consideration.

Keywords: Randić matrix; Spectral decomposition; Star complements

1. Introduction

A graph G considered here is simple, finite and undirected. Denote by $V(G)=\{v_1,\cdots,v_n\}$ the vertex set, $E(G)$ the edge set. The adjacency matrix of G is a $n\times n$ matrix A whose (i,j)-entry is 1 if v_i is adjacent to v_j, and 0 otherwise. The degree of v_i, denoted by d_i, is the number of edges that incident to v_i. The Randić matrix (short for R-matrix) of a graph G is a symmetric matrix $R=(r_{ij})$ whose (i,j)-entry is equal to $1/\sqrt{d_i d_j}$ if v_i is adjacent to v_j, and 0 otherwise. The R-eigenvalues of a graph G are the eigenvalues of its Randić matrix R. One can refer to [2] and [3] for more details about Randić matrix and R-eigenvalues.

In this paper, we give the spectral decomposition of the Randić matrix of graphs, and parallel explant the star set and star complements to the R-eigenvalues. Along with some related results of adjacency eigenvalues, we proof the properties of R-star set and R-star complements of a graph G. Finally, we give a relation between spectral decomposition and star complements of Randić matrix of G, and some further consideration.

2. The Spectral Decomposition of Randić Matrix

Let $e_1,e_2,\cdots,e_n$ be the standard orthonormal basis of R^n. For a graph G, let R be the Randić matrix of G, and $\rho_1>\rho_2>\cdots>\rho_m$ all the distinct eigenvalues of R. Since R is a real symmetric matrix of G, then there exists an orthogonal matrix U such that

$$U^TRU=\begin{pmatrix}\rho_1 & & & \\ & \rho_2 & & \\ & & \ddots & \\ & & & \rho_n\end{pmatrix} \quad (1)$$

For a fixed i, if eigenspace $\varepsilon(\rho_i)$ has an orthonormal basis $y_{i1},y_{i2},\cdots,y_{ik_i}$ for $i=1,\cdots,m$, then set $U=[y_{11},\cdots,y_{1k_1},\cdots,y_{m1},\cdots,y_{mk_m}]$. Thus, we have

$$U^TRU=\begin{pmatrix}\rho_1 I_{k_1} & & & \\ & \rho_2 I_{k_2} & & \\ & & \ddots & \\ & & & \rho_m I_{k_m}\end{pmatrix}=\rho_1\begin{pmatrix}I_{k_1} & & \\ & O & \\ & & \ddots \\ & & & O\end{pmatrix}+\cdots+\rho_m\begin{pmatrix}O & & \\ & O & \\ & & \ddots \\ & & & I_{k_m}\end{pmatrix} \quad (2)$$

Furthermore,

$$R=\rho_1 U\begin{pmatrix}I_{k_1} & & \\ & O & \\ & & \ddots \\ & & & O\end{pmatrix}U^T+\cdots+\rho_m U\begin{pmatrix}O & & \\ & O & \\ & & \ddots \\ & & & I_{k_m}\end{pmatrix}U^T \quad (3)$$

Thus, we have following result.

2.1. Let R be the Randić matrix of a graph G, then R has the spectral decomposition

$$R=\rho_1 U_1+\rho_2 U_2+\cdots+\rho_m U_m \quad (4)$$

For $i=1,\cdots,m$,

$$U_i=U\begin{pmatrix}\ddots & & \\ & I_{k_i} & \\ & & \ddots\end{pmatrix}U^T=y_{i1}y_{i1}^T+\cdots+y_{ik_i}y_{ik_i}^T, \quad (5)$$

where $y_{i1},y_{i2},\cdots,y_{ik_i}$ are the orthonormal basis of $\varepsilon(\rho_i)$. Moreover, $\sum_{i=1}^m U_i=UIU^T=I$, and $U_i^2=U_i=U_i^T$, $i=1,\cdots,m$.

It is straightforward to verify the following result.

2.2. For any polynomial $f(x)=x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n$, we have

$$f(R)=R^n+a_1R^{n-1}+\cdots+a_{n-1}R+a_nI \tag{6}$$

$$=(\rho_1{}^nU_1+\cdots+\rho_m{}^nU_m)+a_1(\rho_1{}^{n-1}U_1+\cdots+\rho_m{}^{n-1}U_m)+\cdots+a_{n-1}(\rho_1U_1+\cdots+\rho_mU_m)+a_nI \tag{7}$$

$$=f(\rho_1)U_1+\cdots+f(\rho_m)U_m \tag{8}$$

In particular, $U_i=f_i(R)$ is a polynomial in R for each i, i.e., $f_i(x)=\prod_{s\neq i}(x-\rho_s)\Big/\prod_{s\neq i}(\rho_i-\rho_s)$.

3. R -Sta Set and R -StarComplements

Let G be a graph with vertex set $V(G)=\{v_1,\cdots,v_n\}$ and Randić matrix R, S be a subset of $V(G)$ such that $|S|<|V(G)|$. The matrix R_S is defined as the principal submatrix of R corresponding to the rows and columns in S.

Let $e_1,e_2,\cdots,e_n$ be the standard orthonormal basis of R^n and E the matrix which represents the orthogonal project of R^n onto the eigenspace $\varepsilon(\rho)$ of R with respect to $e_1,e_2,\cdots,e_n$. Since $\varepsilon(\rho)$ is spanned by the vectors $E_\rho e_j$ $(j=1,2,\cdots,n)$, there exists $X\subseteq V(G)$ such that the vectors $E_\rho e_j$ ($j\in X$) form a basis for $\varepsilon(\rho)$. Such a subset X of $V(G)$ is called a star set for ρ in G.

If X is a star set for ρ in G, then $H=G-X$ is called a star complement for ρ, and $\bar{X}=V(H)=V(G)-X$.

Proposition 3.1. Let G be a graph with ρ as a R-eigenvalue of multiplicity k > 0. Then following conditions on a subset X of $V(G)$ are equivalent:

X is a star for ρ;

$R^n=\varepsilon(\rho)\oplus\varepsilon^0$, where $\varepsilon_0=<e_i:i\notin X>$;

$|X|=k$ and ρ is not an eigenvalue of $R_{\bar{X}}$.

Proposition 3.2. Let $V(G)=\{v_1,\cdots,v_n\}$, and R be the Randić matrix of G. Let E_ρ be defined as above. Then the subset X of $V(G)$ is a star set for ρ in G if and only if the vectors $E_\rho e_i$ $(i\in X)$ form a basis for $\varepsilon_R(\rho)$. Furthermore, the matrix E_ρ is a polynomial function of R, and we have

$$\rho E_\rho e_v=RE_\rho e_v=E_\rho \mathrm{Re}_v=E_\rho\sum_{i\sim v}\frac{1}{\sqrt{d_id_v}}e_i \tag{9}$$

where the summation goes over all vertices that adjacent to vertex v.

Proposition 3.3. Let ρ be a non-zero eigenvalue of Randić matrix R of a connected graph G, and let $K\subset V(G)$ be a subset such that G_K be a connected induced subgraph of G. If R_K does not have ρ as a Randić eigenvalue, then G has a connected star complement for ρ containing K.

Theorem 3.1. Let X be a set of k vertices in the graph G and suppose that G has Randić matrix $\begin{pmatrix} R_X & B^T \\ B & R_{\bar{X}}\end{pmatrix}$, where R_X and $R_{\bar{X}}$ are defined as above. Then X is a star set for ρ in G if and only if ρ is not an eigenvalue of $R_{\bar{X}}$ and $\rho I-R_X=B^T(\rho I_{n-k}-R_{\bar{X}})^{-1}B$.

Proof. Suppose first that X is a star set for ρ. Then ρ is not an eigenvalue of $R_{\bar{X}}$ from Proposition 3.1, and we have

$$\rho I-R=\begin{pmatrix} \rho I-R_X & -B^T \\ -B & \rho I-R_{\bar{X}}\end{pmatrix} \tag{10}$$

where $\rho I-R_{\bar{X}}$ is invertible. In particular, if $|V(G)|=n$, then the matrix $\left(-B\middle|\rho I-R_{\bar{X}}\right)$ has rank $n-k$; but $\rho I-R$ also has rank $n-k$, so the rows of $\left(-B\middle|\rho I-R_{\bar{X}}\right)$ form a basis for the row space of $\rho I-R$. Hence there exists a k × (n − k) matrix L such that

$$\left(\rho I-R_X\middle|-B^T\right)=L\left(-B\middle|\rho I-R_{\bar{X}}\right) \tag{11}$$

Now $\rho I-R_X$ = −LB, −BT = L($\rho I-R_{\bar{X}}$) and the equation follows by eliminating L.

Conversely, if ρ is not an eigenvalue of $R_{\bar{X}}$, then $rank(\rho I-R_{\bar{X}})=n-k$, and $rank(\rho I-R)$ $\geq n-k$, that is $\dim\varepsilon_R(\rho)\leq k$. Let $Y_K\in R^K\setminus\{0\}$, since

$$\begin{pmatrix} \rho I-R_X & -B^T \\ -B & \rho I-R_{\bar{X}}\end{pmatrix}\begin{pmatrix} Y_K \\ (\rho I-R_{\bar{X}})^{-1}BY_K\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \tag{12}$$

then there are at least k linear independent vectors $\begin{pmatrix} Y_K \\ (\rho I-R_{\bar{X}})^{-1}BY_K\end{pmatrix}$ form the eigenvectors of ρ in G, and $\dim\varepsilon_R(\rho)\geq k$. Thus $\dim\varepsilon_R(\rho)=k$, X is a star set for ρ in G from Proposition 3.1.

Theorem 3.2. If X is a star set for ρ in G and $\bar{X}=V(G)-X$, if $\rho\neq 0$ or $\frac{-1}{d_u}$ or $\frac{-1}{d_v}$, where u,v are two vertices with same degree in X, then the $\bar{X}$-neighbourhoods of vertices in X are non-empty and distinct.

Proof. From Proposition 3.2 we have $\rho E_\rho e_u=\sum_{i\sim u}\frac{1}{\sqrt{d_id_u}}E_\rho e_i$. We know from this equation that the vectors in $\{E_\rho e_u\}\cup\{E_\rho e_i:i\sim u\}$ are linear dependent. Since the vectors $E_\rho e_j$ $(j\in X)$ are linear independent, it follows that there is a vertex adjacent to u which lies outside X.

Let $\Gamma(u)$, $\Gamma_X(u)$ and $\Gamma_{\bar{X}}(u)$ be the set of neighboursof u in G, X and $G-X$, respectively. Suppose by way of contradiction that u and v are vertices in X with the same neighbourhoods in $\bar{X}$. From Proposition 3.2 we have

$$\rho E_\rho e_u = \sum_{i\in\Gamma_X(u)} \frac{1}{\sqrt{d_i d_u}} E_\rho e_i + \sum_{i\in\Gamma_{\bar{X}}(u)} \frac{1}{\sqrt{d_i d_u}} E_\rho e_i \quad (13)$$

$$\rho E_\rho e_v = \sum_{j\in\Gamma_X(v)} \frac{1}{\sqrt{d_j d_v}} E_\rho e_j + \sum_{j\in\Gamma_{\bar{X}}(v)} \frac{1}{\sqrt{d_j d_v}} E_\rho e_j \quad (14)$$

Since $d_u \neq 0$, from Eq. (1) we obtain that

$$\sum_{i\in\Gamma_{\bar{X}}(u)} \frac{1}{\sqrt{d_i}} E_\rho e_i = \sqrt{d_u}\rho E_\rho e_u - \sum_{i\in\Gamma_X(u)} \frac{1}{\sqrt{d_i}} E_\rho e_i \quad (15)$$

and also obtain that

$$\sum_{j\in\Gamma_{\bar{X}}(v)} \frac{1}{\sqrt{d_j}} E_\rho e_j = \sqrt{d_v}\rho E_\rho e_v - \sum_{j\in\Gamma_X(v)} \frac{1}{\sqrt{d_j}} E_\rho e_j \quad (16)$$

from Eq. (2). By subtracting the both sides of equations, we have

$$\sqrt{d_u}\rho E_\rho e_u - \sqrt{d_v}\rho E_\rho e_v - \sum_{i\in\Gamma_X(u)} \frac{1}{\sqrt{d_i}} E_\rho e_i + \sum_{j\in\Gamma_X(v)} \frac{1}{\sqrt{d_j}} E_\rho e_j = 0 \quad (17)$$

This is a relation on vectors in $\{E_\rho e_j : j \in X\}$. Since these vectors are linear independent, it follows that either (a) $\rho = 0$, u is not adjacent to v and $\Gamma_X(u) = \Gamma_X(v)$, or (b) $\rho = -\frac{1}{d_u} = -\frac{1}{d_v}$, $u \sim v$ and $\Gamma_X(u)\bigcup\{u\} = \Gamma_X(v)\bigcup\{v\}$, contrary to the assumption.

Theorem 3.3. Suppose that G has ρ as a R-eigenvalue of multiplicity k. If X is a star set for ρ in G and if S is a proper subset of X, |S| = s, then $R_{V(G)-S}$ has ρ as an eigenvalue of multiplicity k − s.

Proof. Since X is a star set for ρ, then from Proposition 3.1 we have $|\rho I - R_{\bar{X}}| \neq 0$. We distinguish three blocks as S, X − S and G − X in matrix $\rho I - R$, one can get that

$$\rho I - R = \begin{pmatrix} \rho I - R_S & -B^T & -E^T \\ -B & \rho I - R_{X-S} & -C^T \\ -E & -C & \rho I - R_{\bar{X}} \end{pmatrix} \quad (18)$$

and

$$\rho I - R_{V(G)-S} = \begin{pmatrix} \rho I - R_{X-S} & -C^T \\ -C & \rho I - R_{\bar{X}} \end{pmatrix}. \quad (19)$$

Since $rank(\rho I - R_{\bar{X}}) = n-k$, we have $rank(\rho I - R_{V(G)-S}) \geq n-k$. Assume by way of contradiction that $rank(\rho I - R_{V(G)-S}) \geq n-k+i$ for i > 1. Then $rank(\rho I - R) \geq n-k+i$, and so $m_G(\rho) \leq k-i$, a contradiction. So we have $rank(\rho I - R_{V(G)-S}) = n-k$. Thus, $R_{V(G)-S}$ has ρ as an eigenvalue of multiplicity k − s. The result follows.

4. The Relation between Spectral Decomposition and Star Complements

Theorem 4.1. The matrices Ui of spectral decomposition and E_{ρ_i} of star complements are equivalent for i = $1,\cdots,m$.

Proof. The proof is similar to that of adjacency matrix A [1].

5. Further Consideration

Let G be a graph of order n with adjacency matrix A and Randić Matrix R. We have known that if A has n_+, n_0, and n_- positive, zero, and negative eigenvalues, respectively ($n_+ + n_0 + n_- = n$), then R has n_+, n_0, and n_- positive, zero, and negative eigenvalues, respectively. Furthermore, If A has a positive (or negative) eigenvalue with multiplicity k, whether R also has an eigenvalue with the same multiplicity?

By star complements, there exists a star set X for λ in G such that $|X| = k$ and $G-X$ is connected. Let $\bar{X} = V(G) - X$, then we can write

$$\lambda I - A = \begin{pmatrix} \lambda I - A_X & -B^T \\ -B & \lambda I - A_{\bar{X}} \end{pmatrix} \quad (20)$$

in which $|\lambda I - A_{\bar{X}}| \neq 0$, and from Theorem 5.1.7 of [1] we have

$$\lambda I - A_X = B^T(\lambda I - A_{\bar{X}})^{-1}B \quad (21)$$

Correspondingly, we can write R as $\begin{pmatrix} R_X & C^T \\ C & R_{\bar{X}} \end{pmatrix}$, where $R_X = D_X^{-\frac{1}{2}} A_X D_X^{-\frac{1}{2}}$, $R_{\bar{X}} = D_{\bar{X}}^{-\frac{1}{2}} A_{\bar{X}} D_{\bar{X}}^{-\frac{1}{2}}$ and $C = D_{\bar{X}}^{-\frac{1}{2}} B D_X^{-\frac{1}{2}}$, that is R_X and $R_{\bar{X}}$ are the principal sub-matrices of R corresponding to the rows and columns in X and $\bar{X}$ respectively.

The question is that whether we can find a R-eigenvalue ρ of G such that ρ is not the eigenvalue of $R_{\bar{X}}$? If the answer is positive, we need additionally to show that $\rho I - R_X = C^T(\rho I - R_{\bar{X}})^{-1}C$. If so, the ρ-eigenvectors has the form $\begin{pmatrix} Y_k \\ (\rho I - R_{\bar{X}})^{-1}CY_k \end{pmatrix}$ from Theorem 3.1, where

$Y_k \in R^k$. Then ρ is the eigenvalue we required. However, from Eq. (3) we only have

$$D_X^{-\frac{1}{2}}(\lambda I - A_X)D_X^{-\frac{1}{2}} = D_X^{-\frac{1}{2}}B^T D_{\bar{X}}^{-\frac{1}{2}} D_{\bar{X}}^{\frac{1}{2}}(\lambda I - A_{\bar{X}})^{-1} D_{\bar{X}}^{\frac{1}{2}} D_{\bar{X}}^{-\frac{1}{2}} B D_X^{-\frac{1}{2}} = C^T D_{\bar{X}}^{\frac{1}{2}}(\lambda I - A_{\bar{X}})^{-1} D_{\bar{X}}^{\frac{1}{2}} C \quad (22)$$

which gives $\lambda D_X^{-1} - R_X = C^T D_{\bar{X}}(\lambda D_{\bar{X}}^{-1} - R_{\bar{X}})^{-1} D_{\bar{X}} C$. Clearly, $\rho I = \lambda D_X^{-1} = \lambda D_{\bar{X}}^{-1}$ if and only if $d_1 = \cdots = d_k = d_{k+1} = \cdots = d_n = \frac{\lambda}{\rho}$, which implies G is a d-regular graph. In this case, we have $D = dI$, and $R = (dI)^{-\frac{1}{2}} A (dI)^{-\frac{1}{2}} = \frac{1}{d} A$. It implies that $\rho_i = \frac{\lambda_i}{d}$ for $i = 1, \cdots, n$.

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