The Least Squares Solution of Large Overdetermined Linear Equations

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Abstract: Because the least squares solutions of large overdetermined linear equations have large relative errors, the least squares solutions of large overdetermined linear equations are studied. Firstly, according to the matrix characteristic analysis of large overdetermined linear equations, the equations are transformed and calculated. Then PCA was used as the theoretical basis to decompose and calculate the large-scale overdetermined linear equations, and the error of the calculation results was calculated to obtain the estimated value with the minimum error. Finally, the dimensionality of the estimated value is reduced to complete the nonnegative treatment of the solution value, and the processed value is output as the least squares solution, thus realizing the calculation of the least squares solution of the large overdetermined linear equations. The experimental results show that the relative error of the proposed method is less than that of the traditional method, and it can meet the precision requirement of the least square solution of the large overdetermined linear equations.

Keywords: Overdetermined linear equations; Least squares solution; Relative error; Nonnegative treatment

1. Introduction

The least square solutions of overdetermined linear equations play an important role in computational science applied mathematics and engineering. As the core problem of computational science, most of the problems in scientific calculation and practical application can be reduced to the least squares solution of the overdetermined linear equations. Common overdetermined linear equations are Topeliet linear equations, Vandermander linear equations, Hankel linear equations, Kauchy linear equations. In addition, it also includes positive definite matrix, positive stable matrix and non-negative matrix, etc. These overdetermined linear equations are widely used in system monitoring, automatic recognition, signal processing, intelligent analysis and engineering calculation, etc. Therefore, it is very necessary to study the least squares solution of overdetermined linear equations [1]. At present, the methods commonly used to solve the least squares of the superdeterministic linear equations are iteration, triangulation, minimal norm fast algorithm and orthogonalization. In the practical application of these methods, the least squares solution of overdetermined linear equations is achieved by means of the normal equations. In general, the final least squares solution can be accurately calculated, but the calculation process is complex and the calculation amount is a little large. When the coefficient matrix is ill-formed, these traditional methods become more ineffective [2]. Especially when solving the least squares problem of large overdetermined linear equations, the direct solution not only consumes a lot of time and memory, but also has a large relative error. Traditional methods can no longer meet the requirements of solving the least-squares solutions of large overdetermined linear equations, so this paper proposes to study the least-squares solutions of large overdetermined linear equations, and tries to design a new solution method to improve the precision of the least-squares solutions.

2. Design of Least Squares Solution Method for Large Overdetermined Linear Equations

Large overdetermined linear equations are composed of two or more matrices, usually expressed as $AX = C$, in which A is the upper triangular matrix, C is the diagonal matrix, and X is the solution of the equations. Because the matrix in the system is larger, the number of rows in the matrix is greater than the number of columns in the matrix. Moreover, the matrix may be ill-formed, and η is usually used to represent the ill-formed degree of the matrix. Therefore, large overdetermined linear equations may have no solutions, and only the least squares solutions can be sought to replace the real solutions of the equations [3]. Based on the above analysis of the characteristics of large overdeter mined linear equations, a solution method is designed. Firstly, it is necessary to study and analyze the matrix of large overdeter mined linear equations with solution in detail. The matrix is transformed according to its characteristics, then PCA is used to solve the equations, and finally the obtained solution value is processed. The solution with the minimum error is taken as the output of the least square solution of the system. The solution process is described in detail below.

2.1. Calculus of large overdetermined linear systems

In order to simplify the solving process of the least square solution of large overdetermined linear equations and avoid redundant calculation process, the transformation and dimensionality reduction of large overdetermined linear equations should be carried out first [4]. The eigenvalues and singular values of ill-posed matrix in the test system are analyzed and expressed as follows:

$$A=KV=ZLV \quad (1)$$

In formula (1), K is the principal component matrix of large overdetermined linear equations, which is the same as the matrix A. V is the eigenvalue of ill-posed matrix in large overdetermined linear equations; L is the singular value of ill-posed matrix in the system of type Overdetermined linear equations; Z is square matrix [5]. n column vectors in the principal component matrix are selected as the principal components. At this point, the equations can be transformed into:

$$K_1 X = C \quad (2)$$

In formula (2), K_1 represents n column vectors in the principal component matrix. Theoretically, the least squares solution x can be obtained by inverting the non-zero elements of L. Since L is invertible, further linear algebraic calculation of large overdetermined linear equations can be proved:

$$AX - B = EL - B \quad (3)$$

In formula (3), E is the n order diagonal matrix, which is the matrix obtained by principal component matrix K filled with an appropriate number of zero elements. Normally, matrix A is full rank, so diagonal matrix E can be regarded as an identity matrix, which satisfies the following conditions:

$$E=\begin{cases}1, & i \in K_1 \\ 0, & \alpha_i = 0\end{cases} \quad (4)$$

Because a represents the chosen principal component vector. According to formula (3), it is not important which principal component vectors are selected for the least squares solution. However, if the principal component vector is small, the least square solution will have a large overflow error. Therefore, the principal components with large eigenvalues and singular values should be selected to complete the calculus of large overdetermined equations.

2.2. PCA solves large overdetermined linear equations

According to the above calculation results of large overdetermined linear equations, PCA was used for the final calculation of formula (3). Firstly, the conditional number of η is taken as the ill-posed characteristic index of linear equations, which is expressed by the formula as follows:

$$\kappa = \frac{V_{\max}}{V_{\min}} \quad (5)$$

In formula (5), A represents the ill-conditioned number of large overdetermined linear equations; B is the maximum eigenvalue; C is the minimum eigenvalue. PCA selects several principal component vectors with large eigenvalues as the denominator, so the denominator becomes larger and the D value becomes smaller. PCA can deal with the ill condition of large overdetermined linear equations in the process of calculation [6]. Then PCA was used to decompose formula (3) into two small equations:

$$\begin{cases}B = KE \\ E = VX\end{cases} \quad (6)$$

Solve the system of equations (6), and get

$$X = VE \quad (7)$$

Take X_1 as the estimated value of the solution of large overdetermined linear equations, and calculate the error of the estimated value. The calculation formula is as follows:

$$h = \|VX_1 - B\| \quad (8)$$

In formula (8), h is the error value of the solution estimate of large overdetermined linear equations, and the estimate value with the minimum error value is output to complete PCA solution of large overdetermined linear equations.

2.3. Least squares solution value processing

If there is no data acquisition, the least square solution of large overdetermined linear equations is non-negative, and even if there is a non-negative solution due to data acquisition error, the absolute value will not be too large. Therefore, data acquisition error is not considered in this research method. However, since the original linear equations were decomposed in the process of solving the least squares solution above, the final solution value may contain a large negative solution, so it is necessary to treat its solution value. In order to satisfy the non-negative value of the solution, the solution value obtained above is revised as follows:

$$x = \max(X_1, 0) \quad (9)$$

Then, dimension reduction is carried out for matrix B, and the equation obtained is:

$$A(XG) = BG \quad (10)$$

In formula (10), BG represents the principal component matrix of B. Then the final solution value is:

$$x = \max(VG) \quad (11)$$

The non-negative solution value obtained by dimension reduction processing according to formula (11) is output as the least squares solution. To sum up, a complete set of least squares solution methods for large overdetermined linear equations were formed by the calculation of the original large overdetermined linear equations, PCA solution, and finally the non-negative treatment of solution values.

3. The experiment

The experiment takes large overdetermined linear equations as the experimental object. The algorithm applied in the experiment is implemented by software Python and runs on Windows10 operating system. All the data in the experimental process, source code and operation result are managed on software YHF, https://sihsad.com/Hksafwd/PCA-Lsafer-easfgdfins. In the experiment, 30% of the data samples were randomly selected as the experimental samples, and the least squares solution of large overdetermined linear equations was solved by using this design method and the traditional method. In order to ensure the validity of the experimental results, two methods are used to solve the equations of 1000-50000 different matrix order, and the relative errors of the two methods are compared. In order to obtain relatively accurate relative error value, this experiment repeated 100 times for each principal component number, and took the average value of the error as the final relative error. The error adopted the 1 norm of the vector, and the relative error was used to measure the approximation ability of the two methods.

Table 1. Comparison of relative error between design method and traditional method

Matrix order		Design method		Traditional method	
m	n	The relative error	time（s）	The relative error	time（s）
1000	20	0.2647-14	5.654	0.9881-14	15.861
2000	20	0.3658-14	10.647	0.9898-14	26.485
3000	20	0.1564-15	12.154	0.9985-14	32.254
4000	20	0.1658-15	15.954	0.9894-14	48.841
5000	20	0.2364-15	28.458	0.9847-15	67.458
6000	20	0.2687-14	32.164	0.9997-15	89.951
7000	20	0.2694-14	35.481	0.9487-14	112.154
8000	20	0.1568-15	45.684	0.9856-15	156.851
9000	20	0.0947-15	89.458	0.9698-14	185.415
10000	20	0.1136-14	115.481	0.9487-14	245.657
20000	20	0.1264-15	264.95	0.9877-14	569.484
50000	20	0.1563-15	468.94	0.9597-14	1954.944

It can be seen from the above table that the relative error of the proposed method is lower than that of the traditional method, and the time of finding the least square solution is also faster than that of the traditional method. This indicates that the decomposition of the design method in the application process will not affect the accuracy of settlement results. Even if the matrix order reaches 50000, the relative error of the least-squares solution is small and has good stability. It is proved that the proposed method can meet the precision requirement of the least-squares solution of large overdetermined linear equations.

4. Conclusion

Combined with relevant literature, the least squares solution of large overdetermined linear equations is studied, which plays a certain role in reducing the relative error of the least squares solution, and provides an important theoretical basis for solving large overdetermined linear equations. This research has important practical and theoretical value for the least squares solution of large overdetermined linear equations. Due to the limited time and personal ability of this research, although some research achievements have been made in this aspect, there are still some problems with the proposed method. The iterative steps of the solution method have not reached the satisfactory level in the scientific application research, and the convergence speed needs to be improved. In the future, in-depth research in this aspect is needed to promote the application and development of large overdetermined linear equations.

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