The Girth of the Zero-Divisor Graph of the Direct Product of Commutative Rings

Qiong LIU*
Department of Mathematics, Shanghai University of Electric Power, Shanghai, CHINA

Abstract: This electronic document is a “live” template. The various components of your paper [title, text, heads, etc.] are already defined on the style sheet, as illustrated by the portions given in this document. (Abstract) In this paper, we consider the zero-divisor graph of the direct product of commutative rings. Specifically, we give the equivalent characterization of the girth of the zero-divisor graph of $R_1 \times R_2$, when thegirth is 3, 4, or ∞, where each R_i is a commutative ring for $i = 1, 2$.

Keywords: Zero-divisor graph; Diameter; Girth; Commutative Rings

1. Introduction

The concept of zero-divisor graph was first defined and studied for commutative rings by I. Beck in [1] and later modified and further studied by D.F. Anderson and P.S. Livingston in [2]. Since then the interplay between algebraic properties of a ring R and the graph theoretic properties of the zero-divisor graph $\Gamma(R)$ has been studied extensively by many authors, see e.g.[3,4,5]. This provides a new area of research, motivating many new views and new unsolved problems to the classical research fields .

For any commutative ring R, we use $\Gamma(R)$ to denote the zero-divisor graph of R. Recall that the zero-divisor graph $\Gamma(R)$ is a simple, connected and undirected graph with diameter less than four and girth less than five. Recall that the vertices of $\Gamma(R)$ are the nonzero elements of $Z(R)$, and there is an edge between a and b if and only if $a \neq b$ and $ab = 0$. In this paper, we use the symbol $a - b$ to denote that the vertex a and b are adjacent to each other. Recall that a simple graph G is called a refinement of a connected simple graph H if $V(G) = V(H)$ and $a - b$ in H implies $a - b$ in G for all distinct vertices a, b of G. In [2, Theorem 2.5], it was pointed out that for any commutative ring R, $\Gamma(R)$ is a refinement of a star graph if and only if either $R \cong Z_2 \times D$, where D is an integral domain, or $Z(R)$ is an annihilator ideal. For a finite commutative ring R, $\Gamma(R)$ is a refinement of a star graph if and only if either $R \cong Z_2 \times F$, where F is a finite field, or R is a local ring [2, Corollary 2.7]). Recall that a graph G is compelte if any two vertex of G is adjacent to each other. A graph G is called a compelte graph with a thorn, if all end vertex of the gaph is adjacent to one vertex of G, all the end vertex is called the thorn of G.

The main purpose of this paper is to study the girth of the zero-divisor graph of $R_1 \times R_2$, where R_1 and R_2 are all commutative rings. Also, we give the equivalent characterization of the girth of $\Gamma(R_1 \times R_2)$.

Throughout this paper, all rings considered will be commutative rings with identity $1 \neq 0$. For any ring R, let N(R) be its nilradical, Z(R) be its set of zero-divisors, Z(R)* be its set of nonzero elements of Z(R). We denote $B_1 = Z_4, B_2 = Z_2[x]/(x^2)$ for convenience, both satisfy $|B_i| = 4$, and $|Z(B_i)| = 2$, where $i = 1, 2$. We adopt graph theoretic notation from [6].

2. Preliminary

Let us recall two useful results from [5,7,8], which will be needed in section 3.

Lemma 1 Let R be a finite commutative ring. Then the following conditions are equivalent:

(1) $gr(\Gamma(R)) = \infty$;

(2) $\Gamma(R)$ is either a star graph or a two-star graph;

(3) R is isomorphic to one of the following rings:

$Z_4, Z_2[x]/(x^2), Z_8, Z_2[x]/(x^3)$,

$Z_4[x]/(x^2 - 2, 2x), Z_9, Z_3[x]/(x^2)$,

$Z_2 \times F$, $Z_2 \times Z_4$, $Z_2 \times Z_2[x]/(x^2)$.

Remark 1 If R is isomorphic to Z_4, or $Z_2[x]/(x^2)$, then the graph $\Gamma(R)$ is an isolated vertex. If R is isomorphic to $Z_8, Z_2[x]/(x^3)$, or $Z_4[x]/(x^2 - 2, 2x)$, then $\Gamma(R)$ is the star graph $K_{1,2}$. If R is isomorphic to Z_9 or $Z_3[x]/(x^2)$, then the graph $\Gamma(R)$ is the star graph $K_{1,1}$.

If R is isomorphic to $Z_2 \times F$, the star graph $K_{1,q}$, where $q=|F|-1$. If R is isomorphic to $Z_2 \times Z_4$, or $Z_2 \times Z_2[x]/(x^2)$, then the zero-divisor graph $\Gamma(R)$ is a two-star graph.

Lemma 2 Let R be a finite commutative ring. Then the following conditions are equivalent:

(1) $gr(\Gamma(R))=3$;

(2) $\Gamma(R)$ is a complete graph K_3 with n thorns, where $n=1,3$;

(3) R is isomorphic to one of the following rings:

$Z_2[x,y]/(x,y)^2, Z_4[x]/(2,x)^2$, $F_4[x]/(x^2), Z_4[x]/(x^2+x+1)$, $Z_2[x]/(x)^4, Z_4[x]/(x^2+2), Z_4[x]/(x^2+2x+2)$, $Z_4[x]/(x^3-2,2x), Z_{16}, Z_2 \times Z_2 \times Z_2$, $Z_2[x,y]/(x^2,y^2), Z_4[x]/(x^2)$, $Z_4[x,y]/(x^2,y^2,xy-2)$.

Remark 2 If R is isomorphic to $Z_2[x,y]/(x,y)^2$, $Z_4[x]/(2,x)^2$, $F_4[x]/(x^2)$, or $Z_4[x]/(x^2+x+1)$, then the graph $\Gamma(R)$ is the complete graph K_3. If R is isomorphic to $Z_2[x]/(x)^4, Z_4[x]/(x^2+2)$, $Z_4[x]/(x^2+2x+2)$, $Z_4[x]/(x^3-2,2x)$, or Z_{16}, then the graph $\Gamma(R)$ is a complete graph K_3 with a thorn. If R is isomorphic to $Z_2 \times Z_2 \times Z_2$, then the graph $\Gamma(R)$ is the triangle with three thorns. If R is isomorphic to $Z_2[x,y]/(x^2,y^2), Z_4[x]/(x^2)$, or $Z_4[x,y]/(x^2,y^2,xy-2)$, then the graph $\Gamma(R)$ is the fan graph F_3 with three leaves.

Lemma 3 Let R be a finite commutative ring. Then the following conditions are equivalent:

(1) $gr(\Gamma(R))=4$;

(2) $\Gamma(R)$ is a complete bipartite graph or a complete bipartite graph with a thorn;

(3) R is isomorphic to one of the following rings: $D_1 \times D_2, B_i \times D (i=1,2)$.

Remark 3 If R is isomorphic to $D_1 \times D_2$, then the zero-divisor graph $\Gamma(R)$ is a complete bipartite graph. If R is isomorphic to $B_i \times D$, then the zero-divisor graph $\Gamma(R)$ is a complete bipartite graph with a thorn for each $i=1,2$.

3. Main Results

Now we give the main result of this paper in this section.

Theorem 1 Let R_1 and R_2 be commutative rings. Then $gr(\Gamma(R_1 \times R_2))=4$ if and only if R_1 is domain with $|R_1|\geq 3$, and R_2 is either domain with $|R_2|\geq 3$, or $R_2 \cong B_i (i=1,2)$.

Proof "$\Leftarrow$". Clearly.

"$\Rightarrow$". Let $G=\Gamma(R_1 \times R_2)$. If $gr(G)=4$, then G contains no triangles. By Lemma 3, G is either a complete bipartite graph or a complete bipartite graph with a thorn. If G is a complete bipartite graph. We claim that each R_1 is domain $(i=1,2)$. In fact, if $Z(R_1)\neq 0$, then there exist $a_1, b_1 \in Z(R_1)^*$ such that $a_1 b_1 = 0$. Let $G=V_1 \cup V_2$, where V_1, V_2 are two vertex set of G. From $(0,1)-(a_1,0)-(b_1,1)$, we can assume $(a_1,0)\in V_1$ and $(b_1,1),(0,1)\in V_2$. $(0,1)$ and $(1,0)$ are adjacent implies $(1,0)\in V_1$. Thus $(1,0)$ and $(b_1,1)$ are adjacent, and hence $b_1=0$. A contradiction. If G is a complete bipartite graph with a thorn, then G is complemented, but not uniquely complemented. By Lemma 1, we can see that R_1 is domain and R_2 is isomorphic to $B_i (i=1,2)$.

Corollary 1 For any finite commutative ring R, $gr(\Gamma(R))=4$ if and only if either $R \cong F_1 \times F_2$ for finite fields F_i with $|F_i|\geq 3 (i=1,2)$, or $R \cong F \times B_i$ for finite field F with $|F|\geq 3$.

Proof If R is local with maximal ideal M, then Z(R)=M=Ann(x) for some $x \in Z(R)^*$. This implies that $\Gamma(R)$ is a refinement of a star graph, and hence $gr(\Gamma(R))$ is either infinite or less than 4. This is impossible. Therefore, R is a finite direct product of Artinian local rings. Write R as $R_1 \times R_2 \times ... \times R_n$ $(n\geq 2)$, where each R_i is Artinian local. Notice that if $n\geq 3$, then there is a triangle $(1,0,0,...)-(0,1,0,...)-(0,0,1,...)-(1,0,0,...)$.

This is impossible since $gr(\Gamma(R))=4$. Thus $n=2$ and $R \cong R_1 \times R_2$ for finite local rings $R_i (i=1,2)$. The rest is directly from Theorem 1 and Lemma 2.

Theorem 2 Let R_1 and R_2 be commutative rings. Then $gr(\Gamma(R_1 \times R_2))=3$ if and only if either $R_1 \cong B_i$ $(i=1,2)$ and R_2 is not domain, or $|Z(R_1)|\geq 3$ and R_2 is any arbitrary commutative ring.

Proof We only need to prove the necessary condition. Consider the value of $|Z(R_i)| (i=1,2)$. Clearly, $|Z(R_i)|\geq 1$ for all $i=1,2$. By Theorem 2, R_1 and R_2 can not be both domains with $|R_i|\geq 3$. Assume that R_1 is not domain. Then $|Z(R_1)|\geq 2$. If $|Z(R_1)|=2$, then $R_1 \cong B_i (i=1,2)$. By Theorem 1, R_2 is not domain. If

$|Z(R_1)|\geq 3$, then for any commutative ring R_2, there exists a triangle

$$(a_1,0)-(a_2,0)-(0,1)-(a_1,0),$$

where $a_1,a_2\in Z(R_1)^*$ and $a_1\neq a_2$. Thus the girth of the zero-divisor graph of $R_1\times R_2$ is three.

Lemma 4 Let R be a commutative ring and R is not domain. Then $gr(\Gamma(B_i\times R))=3(i=1,2)$.

Proof We only need consider the case $B_1=Z_4$. By assumption, $|Z(R)|^*\geq 1$. If $|Z(R)|^*=1$, let $a\in Z(R)^*$ such the $a^2=0$. Then

$$(\overline{2},a)-(\overline{0},a)-(\overline{2},0)-(\overline{2},a)$$

forms a triangle and hence the girth is three. If $|Z(R)|^*\geq 2$, we can prove similarly. Also, we can prove $gr(\Gamma(B_2\times R))=3$ in the same way.

Theorem 3 For any commutative ring and R_1 and R_2, $gr(\Gamma(R_1\times R_2))=\infty$ if and only if $R_1\cong Z_2$ and either R_2 is domain or $R_2\cong B_i$ $(i=1,2)$.

Proof We only need to prove the necessary condition. Clearly, $|\Gamma(R_1\times R_2)|\geq 2$. Therefore, $\Gamma(R_1\times R_2)$ is either a star graph or a two-star graph. If $|R_i|>2$ for all $i=1,2$, then there is a rectangle

$$(a_1,0)-(0,1)-(1,0)-(0,a_2)-(a_1,0),$$

where $a_i\in R_i\setminus\{0,1\}(i=1,2)$. This is impossible since $gr(\Gamma(R_1\times R_2))=\infty$. Thus either $|R_1|=2$ or $|R_2|=2$. Assume that $|R_1|=2$. Then $R_1\cong Z_2$. If $|Z(R_2)|\geq 3$, we can check that there is a triangle in $\Gamma(R_1\times R_2)$. Thus $|Z(R_2)|\leq 2$. If $|Z(R_2)|=1$, then R_2 is domain, and hence $\Gamma(R_1\times R_2)$ is a star graph. If $|Z(R_2)|=2$, then $R_2\cong B_i$ $(i=1,2)$ by Lemma 1. Therefore, $\Gamma(R_1\times R_2)$ is a two-star graph and by Lemma 1, we can easily get the result.

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