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The Chromatic Number of Zero-divisor Graphs (I)

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Abstract: In this paper, we discuss the chromatic number of the zero-divisor graphs $\Gamma(R)$. We study the algebraic structure and properties of commutative rings whose zero-divisor graphs have its chromatic number equals one or two. Furthermore, we completely determine the algebraic structure of rings R.

Keywords: Commutative rings; Star graphs; Two-star graphs; Bipartite graphs.

1. Introduction

The main purpose of this paper is to present the ideal of coloring of a commutative ring. This idea establishes the connections between graph theories and commutative ring theories which can be turn out to be mutually beneficial for these two branches of mathematics. In this article, we will be interested in characterizing and discussing the rings which are one or two colorable, that is, finitely colorable. Then, we will use possible applications to graph theories.

Throughout this paper, we suppose R is a commutative ring and we consider $\Gamma(R)$ as a simple graph whose vertices are the elements of $Z(R)^*$, and there is an edge between x and y if and only if $xy = 0$ and $x \neq y$. This concept of a zero-divisor graph for commutative rings was first defined and studied in [1] by Beck. Later, it was modified and further studied by many authors, e.g. [2,3,4,5]. In [2], DeMeyer, Mckenzie and Schneider began the study of zero-divisor graph of a commutative semigroup with 0. Since then, a lot of fundamental results on the concept have been established in [6,7]. Recall that the zero-divisor graph $\Gamma(R)$ is a simple, connected and undirected graph with diameter less than four and girth less than five. The core of $\Gamma(R)$ is either empty or a union of triangles and rectangles. For a commutative ring R with $\Gamma(R)$ nonempty, the graph $\Gamma(R)$ is finite if and only if R is a finite ring. Recall that a simple graph G is called a refinement of a connected simple graph H if $V(G) = V(H)$ and the vertex x is adjacent to y in H implies the vertex x is adjacent to y in G. Recall that a graph G is compelte if any two vertex of G is adjacent to each other.

In this paper, we let ϕ(R) to denote the chromatic number of the graph $\Gamma(R)$, i.e., the minimal number of colors which can be assigned to be the elements of R in such a way that every two adjacent elements have different colors.

A subset C={x1,…, xn} is called a clique provided $x_i x_j$ for all $i \neq j$. If R contains a clique with n elements, and every clique has at most n elements, we say that the clique number of R is n and write Clique(R)=n. If the sizes of the cliques in R are not bounded we define Clique(R)=∞. Clearly, the case Clique(R)=∞ entails the existence of an infinite clique.

Clearly, ϕ(R)≥ Clique(R) and for most graphs G we certainly have ϕ(R)> Clique(R).

The main purpose of this paper is to study the algebraic structure rings R by taking advantage of its zero-divisor graphs $\Gamma(R)$. Throughout this paper, we adopt graph theoretic notations from [9]. For notions and results about commutative commutative rings, we use [8] as basic references. We use the symbol $x - y$ to denote that the vertices x and y are adjacent. As in [8], we define the neighborhood N(a) of a vertex a to be the set of all vertices which are adjacent to a. We adopt graph theoretic notation from [10]. All rings considered will be commutative rings with identity $1 \neq 0$. For any ring R, let N(R) be its nilradical, Z(R) be its set of zero-divisors, Z(R)* be the set of all nonzero elements of Z(R). For a graph G, we say a ring R is a corresponding ring of G if $\Gamma(R) \cong G$.

Throughout this paper, we discuss the chromatic number of the zero-divisor graphs $\Gamma(R)$. We will study the algebraic structure and properties of commutative rings whose zero-divisor graphs have its chromatic number equals one or two.

2. Preliminary

Let us recall some useful results about zero-divisor graphs, which will be needed in the Section 3.

Lemma 1 Let R be a finite commutative ring. Then ϕ(R)=1 if and only if $\Gamma(R)$ is an isolated vertex.

Proof "$\Leftarrow$". If $\Gamma(R)$ is an isolated vertex, then he chromatic number ϕ(R) of the graph $\Gamma(R)$ is one.

"$\Rightarrow$". If ϕ(R)=1, then $\Gamma(R)$ is a vertex set in which each vertex set are not adjacent to each other. Since $\Gamma(R)$ is connected, it implies that $\Gamma(R)$ must be an isolated vertex.

Lemma 2 Let R be a finite commutative ring. Then ϕ(R)=2 if and only if $\Gamma(R)$ is a star graph, a two-star graph, or a bipartite graph.

Proof "$\Leftarrow$". If $\Gamma(R)$ is a star graph, let c be the center of $\Gamma(R)$ and $\{x_i | i \geq 1\}$ be the end vertices of $\Gamma(R)$. Clearly, c is adjacent to $\{x_i | i \geq 1\}$ and the vertices in $\{x_i | i \geq 1\}$ are not adjacent to each other. Then we can color the vertices set $\{x_i | i \geq 1\}$ with one color and color the vertex c with another color. Thus the chromatic number ϕ(R) of the graph $\Gamma(R)$ is two.

If $\Gamma(R)$ is a two-star graph, let c, c' be the two centers of $\Gamma(R)$, let $\{x_i | i \geq 1\}$ be the end vertices adjacent to c, and $\{y_i | i \geq 1\}$ be the end vertices adjacent to c'. Then we can color the vertex set $\{c, y_i | i \geq 1\}$ with one color and color the vertex $\{c', x_i | i \geq 1\}$ with another color. This implies that the chromatic number ϕ(R) of the two-star graph $\Gamma(R)$ is two.

If $\Gamma(R)$ is a bipartite graph, let V, V' be the two vertex sets of $\Gamma(R)$. It is easy to see that every vertices in V or V' are not adjacent to each other. So we can color the vertex set V with one color and color the other vertex set V' with another color. It follows that the chromatic number ϕ(R) of the bipartite graph is two.

"$\Rightarrow$". If the chromatic number ϕ(R) of the bipartite graph is two, then let V, V' be the two vertex sets of $\Gamma(R)$. It is easy to find that the vertices in V are not adjacent to the vertices in V', and every vertices in V or V' are not adjacent to each other. This implies that the graph $\Gamma(R)$ is a star graph if |V|=1, $\Gamma(R)$ is a two-star graph if |V|=2, $\Gamma(R)$ is a bipartite graph if |V|≠1,2.

3. Main Results

In the following, we will discuss the algebraic structure of rings R whose chromatic number is one or two.

Theorem 1 Let R be a finite commutative ring. If ϕ(R)=1, then R is isomorphic to Z_4 or $Z_2[x]/(x^2)$.

Proof By Lemma 1, if $\Gamma(R)$ is a star graph, we can assume $Z(R)=\{0, x\}$, where $x \neq 0$ is the only one vertex of $\Gamma(R)$. Then we have the following exact sequence

$0 \to Ann(x) \to R \to Rx \to 0$.

It is easy to get that $R/Ann(x) \cong Rx$, where $Rx = Ann(x) = \{0, x\}$. Therefore, $|R| = 4$. Now, we consider the characteristic of R. Clearly, $char(R)|4$. If $char(R)=4$, then $R \cong Z_4$. If $char(R)=2$, then $R \cong Z_2[x]/(x^2)$.

Theorem 2 Let R be a finite commutative ring. If ϕ(R)=2, then R is isomorphic to one of the following rings:

$$Z_2 \times Z_4,\ Z_2 \times Z_2[x]/(x^2),\ Z_2 \times D,\ D_1 \times D_2,$$

where D, D1, D2 are integral domains.

Proof By Lemma 2, let us assume $\Gamma(R)$ is a two-star graph, there exist one path $x-y-z-w$ of length 3, and the two path $x-y-z$ and $y-z-w$ are not contained in any cycle. It is clearly that $R/Ann(y) \cong Z_2$ and $R/Ann(z) \cong Z_2$. This follows that the two ideals $Ann(y), Ann(z)$ are the maximal ideals of R, and therefore $Ann(y) \neq Ann(z)$.

Now,let $I = Ann(y) \cap Ann(z)$. Clearly, $I \neq \{0\}$. Since the path $y-z$ is not contained in any cycle, we obtain $I \subseteq \{0, y, z\}$. If $I = \{0, y, z\}$, then $y+z=0$ and therefore $y=-z$. We can conclude that $Ann(y)=Ann(z)$, a contradiction. Thus either $I=\{0,y\}$ or $I=\{0,z\}$. By symmetry, assume $I=\{0,y\}$. Then $y^2=0, z^2 \neq 0$. Consider the value of z^2. If $z^2 \neq z$, let $z^2 = y$, then $yw=0$ and there is a triangle $y-z-w-y$. A contradiction. Also, we have $z^2 \neq w$. Since $z^2 w = 0 = z^2 y$, there is a rectangle $y-z-w-z^2-y$. Impossible. Therefore, $z^2 = z$. Then, we get the direct decomposition $R = Rz \oplus R(1-z)$. Let us consider $R(1-z)$. Since $R/J \cong R/Ann(y) \times R/Ann(z)$, we have $R/\{0,y\} \cong Z_2 \times Z_2$ and $|R|=8$. By $R(1-z) \cong R/Rz$, we can find that $|R(1-z)|=4$. It is easy to check that in this case, $R(1-z)$ is isomorphic to Z_4 or $Z_2[x]/(x^2)$. Hence, R is isomorphic to $Z_2 \times Z_4$ or $Z_2 \times Z_2[x]/(x^2)$.

If $\Gamma(R)$ is a bipartite graph, let V, V' be the two vertex sets of $\Gamma(R)$. It is easy to see that the vertices in V are not adjacent to the vertices in V', and every vertices in V or V' are not adjacent to each other. If |V|=1, then $\Gamma(R)$ is a star graph and hence $R \cong Z_2 \times D$, If |V|≠1 and |V' |≠1, then $R \cong D_1 \times D_2$, where D1, D2 are integral domains.

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