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Efficient Domination in Cayley Graphs

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Abstract: An E-chain is a countable family of nested graphs, each of which has an efficient dominating set. In this paper, we construct two E-chains of circulant graphs, which are Cayley graphs on finite cyclic groups. In addition, we give a necessary and sufficient condition for the existence of efficient dominating sets in any Cayley graphs.

Keywords: Circulant graph; Cayley graph; Efficient dominating set

1. Introduction

For a simple graph Γ, we denote its vertex set and edge set respectively by $V(\Gamma)$ and $E(\Gamma)$. Let G be a finite group with e as the identity and S be a subset of G not containing e such that $S=S^{-1}$. The Cayley graph $Cay(G,S)$ on G with respect to S is defined as the graph with vertex set $V(Cay(G,S))=G$ and edge set $E(Cay(G,S))=\{\{g,sg\}:g\in G, s\in S\}$. Clearly, $Cay(G,S)$ is a $|S|-$ regular graph, and $Cay(G,S)$ is connected if and only if G is generated by S. A Cayley graph on finite cyclic group $\mathbf{Z}_n:=\{0,1,\ldots,n-1\}$ with addition modulo n is called a circulant graph.

A subset I of vertices in a graph Γ is called an independent set if no two vertices in I are adjacent. A maximum independent set of Γ is an independent set of largest possible size, and this size, denoted by $\alpha(\Gamma)$, is called the independence number of Γ. A subset D of vertices in a graph Γ is called a dominating set if each vertex not in D is adjacent to at least one vertex in D. The domination number of a graph Γ is the minimum size of a dominating set of Γ, and denoted by $\gamma(\Gamma)$. A subset D of vertices in a graph Γ is called an efficient dominating set (also called a perfect code) if D is an independent set and each vertex not in D is adjacent to exactly one vertex in D. A countable family of graphs $\mathbf{F}=\{\Gamma_1\subset\Gamma_2\subset\ldots\subset\Gamma_i\subset\Gamma_{i+1}\subset\ldots\}$ is called an E-chain if every Γ_i is an induced subgraph of Γ_{i+1} and each Γ_i admits an efficient dominating set.

In the past few years efficient dominating sets in Cayley graphs, especially in circulant graphs, have attracted received much attention. For example, Lee [1] proved that a Cayley graph on an abelian group admits an efficient dominating set if and only if it is a covering graph of a complete graph. Dejter and Serra [2] gave a constructing tool to produce infinite families of E-chains of Cayley graphs on symmetric groups. Huang and Xu [3] determined the existence of efficient dominating sets in several classes of circulant graphs. Tamizh Chelvam and Mutharasu [4] constructed an E-chain of circulant graphs. In addition, Tamizh Chelvam and Mutharasu [5] also gave a necessary and sufficient condition for a subgroup to be an efficient dominating set in circulant graphs. Kumar and MacGillivray [6] characterized the efficient dominating sets in circulant graphs with domination number 2 and 3, and Deng [7] further investigated the existence and construction of efficient dominating sets in circulant graphs with domination number prime. Obradović et al. [8] gave necessary and sufficient conditions for the existence of efficient dominating sets in connected circulant graphs of degree 3 and 4. Feng et al. [9] and Deng et al. [10] independently generalized the results in [8] by considering the case of more general degree.

In this paper, we construct two E-chains of circulant graphs, and give a necessary and sufficient condition for the existence of efficient dominating sets in any Cayley graphs.

2. E-chains in Circulant Graphs

2.1. Preliminary lemmas

Let G be a finite abelian group with additive notation. We define the sum of two subsets M,N of G by $M+N=\{m+n: m\in M, n\in N\}$. If each $x\in M+N$ has a unique representation in the form $x=m+n$ with $m\in M$ and $n\in N$, then the sum $M+N$ is called direct, and denoted by $M\oplus N$.

Lemma 2.1 [11] Let G be a finite abelian group and let M,N be subsets of G. Then $G=M\oplus N$ is equivalent to the conjunction of any two of the following conditions:

(i) $G=M+N$;

(ii) $(M-M)\cap(N-N)=\{0\}$;

(iii) $|G|=|M||N|$.

By the definitions of Cayley graph and efficient dominating set, one can obtain the following lemma.

Lemma 2.2 [7] Let S be a subset of a finite abelian group G not containing 0 such that $S=-S$, and let $S_0 = S \cup \{0\}$. Then $Cay(G,S)$ admits an efficient dominating set D if and only if $G = S_0 \oplus D$.

2.2. Constructing one E-chain

Lemma 2.3 Let n,m be positive integers and let M,N be subsets of $\mathbf{Z}_n$ .If $\mathbf{Z}_n = M \oplus N$, then $\mathbf{Z}_{nm} = (M + n\mathbf{Z}_m) \oplus N$.

Proof. Since $\mathbf{Z}_n = M \oplus N$, it follows from Lemma 2.1 that $\mathbf{Z}_n = M + N$ and $n = |M||N|$. Therefore, we have $(M + n\mathbf{Z}_m) + N = (M+N) + n\mathbf{Z}_m = \mathbf{Z}_n + n\mathbf{Z}_m = \mathbf{Z}_{nm}$, and $|M + n\mathbf{Z}_m||N| = (|M|m)|N| = (|M||N|)m = nm$, which together with Lemma 2.1 implies that $\mathbf{Z}_{nm} = (M + n\mathbf{Z}_m) \oplus N$. The assertion holds.

Theorem 2.4 Let Γ_i be the circulant graphs $Cay(\mathbf{Z}_{n_1 n_2 \cdots n_i}, S_i)$ for $i = 1,2,\cdots$ and D be an efficient dominating set of Γ_1, where S_1 is a subset of $\mathbf{Z}_{n_1} \setminus \{0\}$ such that $S_1 = -S_1$, and $S_{i+1} = \left(\left(S_i \cup \{0\}\right) + n_i \mathbf{Z}_{n_{i+1}}\right) \setminus \{0\}$ for $i = 1,2,\cdots$. Then the family $\mathbf{F} = \{\Gamma_i : i = 1,2,\cdots\}$ is an E-chain and D is an efficient dominating set of each $\Gamma_i \in \mathbf{F}$.

Proof. First it is easy to see that each Γ_i is an induced subgraph of Γ_{i+1}. Therefore, it is suffices to show that D is an efficient dominating set of each $\Gamma_i \in \mathbf{F}$.

Since D is an efficient dominating set of Γ_1, it follows from Lemma 2.2 that $\mathbf{Z}_{n_1} = \left(S_1 \cup \{0\}\right) \oplus D$. By Lemma 2.3, $\mathbf{Z}_{n_1 n_2} = \left(\left(S_1 \cup \{0\}\right) + n_1 \mathbf{Z}_{n_2}\right) \oplus D = \left(S_2 \cup \{0\}\right) \oplus D$, which together with Lemma 2.2 implies that D is an efficient dominating set of Γ_2. Similarly, we can deduce by analogy that $\mathbf{Z}_{n_1 n_2 \cdots n_i} = \left(S_i \cup \{0\}\right) \oplus D$ for $i = 1,2,\cdots$, and thus D is an efficient dominating set of each $\Gamma_i \in \mathbf{F}$. The assertion holds.

2.3. Constructing another E-chain

Lemma 2.5 Let n,m be positive integers and let M,N be subsets of $\mathbf{Z}_n$. If $\mathbf{Z}_n = M \oplus N$, then $\mathbf{Z}_{nm} = (\mathrm{m}M + \mathbf{Z}_m) \oplus mN$.

Proof. Since $\mathbf{Z}_n = M \oplus N$, it follows from Lemma 2.1 that $\mathbf{Z}_n = M + N$ and $n = |M||N|$. Therefore, we have $(\mathrm{m}M + \mathbf{Z}_m) + mN = m(M+N) + \mathbf{Z}_m = m\mathbf{Z}_n + \mathbf{Z}_m = \mathbf{Z}_{nm}$, and $|\mathrm{m}M + \mathbf{Z}_m||mN| = (|M|\mathrm{m})|N| = (|M||\mathrm{N}|)m = nm$, which together with Lemma 2.1 implies that $\mathbf{Z}_{nm} = (\mathrm{m}M + \mathbf{Z}_m) \oplus mN$. The assertion holds.

Theorem 2.6 Let Γ_i be the circulant graphs $Cay(\mathbf{Z}_{n_1 n_2 \cdots n_i}, S_i)$ for $i = 1,2,\cdots$ and D be an efficient dominating set of Γ_1, where S_1 is a subset of $\mathbf{Z}_{n_1} \setminus \{0\}$ such that $S_1 = -S_1$, and $S_{i+1} = \left(n_{i+1}\left(S_i \cup \{0\}\right) + \mathbf{Z}_{n_{i+1}}\right) \setminus \{0\}$ for $i = 1,2,\cdots$. Then the family $\mathbf{F} = \{\Gamma_i : i = 1,2,\cdots\}$ is an E-chain and $n_i n_{i-1} \cdots n_2 D$ is an efficient dominating set of Γ_i for $i = 2,3,\cdots$.

Proof. For $i = 1,2,\cdots$, defining a mapping f_i as follows: $f_i : \mathbf{Z}_{n_1 n_2 \cdots n_i} \to \mathbf{Z}_{n_1 n_2 \cdots n_{i+1}}, g \mapsto n_{i+1} g$. Then it is easy to see that f_i is a one-to-one graph homomorphism from Γ_i to Γ_{i+1}, and thus Γ_i is an induced subgraph of Γ_{i+1} for $i = 1,2,\cdots$. Therefore, it is suffices to show that $n_i n_{i-1} \cdots n_2 D$ is an efficient dominating set of Γ_i for $i = 2,3,\cdots$.

Since D is an efficient dominating set of Γ_1, it follows from Lemma 2.2 that $\mathbf{Z}_{n_1} = \left(S_1 \cup \{0\}\right) \oplus D$. By Lemma 2.5, $\mathbf{Z}_{n_1 n_2} = \left(n_2\left(S_1 \cup \{0\}\right) + \mathbf{Z}_{n_2}\right) \oplus n_2 D = \left(S_2 \cup \{0\}\right) \oplus n_2 D$, which together with Lemma 2.2 implies that $n_2 D$ is an efficient dominating set of Γ_2. Similarly, we can deduce by analogy that $\mathbf{Z}_{n_1 n_2 \cdots n_i} = \left(S_i \cup \{0\}\right) \oplus n_i n_{i-1} \cdots n_2 D$ for $i = 2,3,\cdots$, and thus $n_i n_{i-1} \cdots n_2 D$ is an efficient dominating set of Γ_i for $i = 2,3,\cdots$. The assertion holds.

3. Efficient Domination in Cayley Graphs

3.1. Preliminary lemmas

In this section, we assume that G is a finite group with multiplicative notation. The product of two subsets M,N of finite group G is defined as $MN = \{mn : m \in M, n \in N\}$. If each $x \in MN$ has a unique representation in the form $x = mn$ with $m \in M$ and $n \in N$, then the product MN is called direct, and denoted by $M \times N$.

Similar to Lemmas 2.1 and 2.2, we have the following two lemmas.

Lemma 3.1 Let G be a finite group with identity element e and let M,N be subsets of G. Then $G = M \times N$ is equivalent to the conjunction of any two of the following conditions:

(i) $G = MN$;

(ii) $M^{-1}M \cap NN^{-1} = \{\mathrm{e}\}$;

(iii) $|G| = |M||N|$.

Lemma 3.2 Let S be a subset of a finite group G not containing the identity e such that $S = S^{-1}$, and let $S_e = S \cup \{\mathrm{e}\}$. Then $Cay(G,S)$ admits an efficient dominating set D if and only if $G = S_e \times D$.

3.2. Independence number and efficient domination

Theorem 3.3 Let S be a subset of a finite group G not containing the identity e such that $S=S^{-1}$, and let $S_e=S\cup\{e\}$. Then $\alpha(Cay(G,S_e^{-1}S_e\bullet\{e\}))\le\frac{|G|}{|S|+1}$.

Proof. We suppose on the contrary that $\alpha(Cay(G,S_e^{-1}S_e\setminus\{e\}))>\frac{|G|}{|S|+1}$. Thus, there exist an independent set I of $Cay(G,S_e^{-1}S_e\setminus\{e\})$ such that $|I|>\frac{|G|}{|S|+1}$. By the definitions of independent set and Cayley graph, it is easy to see that $(S_e^{-1}S_e\bullet\{e\})\cap II^{-1}=\varnothing$, that is, $S_e^{-1}S_e\cap II^{-1}=\{e\}$, which implies that the product of S_eI is direct. Since the direct product $S_e\times I\subseteq G$, it follows that $|S_e||I|\le|G|$, and thus $|I|\le\frac{|G|}{|S|+1}$, which contradicts with $|I|>\frac{|G|}{|S|+1}$. Hence the assertion holds.

Theorem 3.4 Let S be a subset of a finite group G not containing the identity e such that $S=S^{-1}$, and let $S_e=S\cup\{e\}$. Then $\gamma(Cay(G,S))=\frac{|G|}{|S|+1}$ if and only if $\alpha(Cay(G,S_e^{-1}S_e\bullet\{e\}))=\frac{|G|}{|S|+1}$.

Proof. If $\gamma(Cay(G,S))=\frac{|G|}{|S|+1}$, then there exists a dominating set D such that $|D|=\frac{|G|}{|S|+1}$. Note that $Cay(G,S)$ is an $|S|-$ regular graph. It is easy to see that D is an efficient dominating set of $Cay(G,S)$. By Lemmas 3.1 and 3.2, $S_e^{-1}S_e\cap DD^{-1}=\{e\}$, and thus $(S_e^{-1}S_e\setminus\{e\})\cap DD^{-1}=\varnothing$, which implies that D is an independent set of $Cay(G,S_e^{-1}S_e\setminus\{e\})$. Therefore, $\alpha(Cay(G,S_e^{-1}S_e\bullet\{e\}))\ge\frac{|G|}{|S|+1}$, which together with Theorem 3.3 implies that $\alpha(Cay(G,S_e^{-1}S_e\bullet\{e\}))=\frac{|G|}{|S|+1}$.

If $\alpha(Cay(G,S_e^{-1}S_e\bullet\{e\}))=\frac{|G|}{|S|+1}$, then there exist an independent set I of $Cay(G,S_e^{-1}S_e\bullet\{e\})$ such that $|I|=\frac{|G|}{|S|+1}$, which implies that $S_e^{-1}S_e\cap II^{-1}=\{e\}$. By Lemma 3.1, $G=S_e\times I$. By Lemma 3.2, I is an efficient dominating set of $Cay(G,S)$, which implies that $\gamma(Cay(G,S))=\frac{|G|}{|S|+1}$. The assertion holds.

Lemma 3.5 [3] Let Γ be a $k-$regular graph of order n Then Γ admits an efficient dominating set if and only if $\gamma(\Gamma)=\frac{n}{k+1}$.

Combining Theorem 3.4 and Lemma 3.5, one can easily obtain the following result.

Theorem 3.6 Let S be a subset of a finite group G not containing the identity e such that $S=S^{-1}$, and let $S_e=S\cup\{e\}$. Then $Cay(G,S)$ admits an efficient dominating set if and only if $\alpha(Cay(G,S_e^{-1}S_e\bullet\{e\}))=\frac{|G|}{|S|+1}$.

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