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Efficient Dominating Sets in Cayley Graphs on Symmetric Groups

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Abstract: For any given positive integers n and k with $n>k$, let $T^{(n,k)}=\{s_i : i\in\{1,2,\ldots,n\}\setminus\{k\}\}$ be a generating set of the symmetric group S_n of degree n, where each $s_i\in T^{(n,k)}$ satisfies that $k^{s_i}=i, i^{s_i}=k$, and each $s_i\in T^{(n,k)} \bullet \{s_n\}$ satisfies that $n^{s_i}=n$. The Cayley graph on S_n with respect to $T^{(n,k)}$ is denoted by $\Gamma^{(n,k)}$. In this paper, we determine the domination number of $\Gamma^{(n,k)}$, and obtain some results about efficient dominating sets in $\Gamma^{(n,k)}$.

Keywords: Symmetric group; Cayley graph; Efficient dominating set

1. Introduction

For a simple graph Γ, we denote its vertex set and edge set respectively by $V(\Gamma)$ and $E(\Gamma)$. Let G be a finite group with e as the identity and S be an inverse-closed subset of G not containing e. The Cayley graph $Cay(G,S)$ of G with respect to S is defined by $V(Cay(G,S))=G, E(Cay(G,S))=\{\{g,sg\}: g\in G, s\in S\}$. Clearly, $Cay(G,S)$ is a graph of degree $|S|$, and $Cay(G,S)$ is connected if and only if G is generated by S.

A subset D of vertices in a graph Γ is called a dominating set if each vertex not in D is adjacent to at least one vertex in D. The domination number of a graph Γ is the minimum size of a dominating set of Γ, and denoted by $g(\Gamma)$. A subset D of vertices in a graph Γ is called an efficient dominating set (also called a perfect code) if D is an independent set and each vertex not in D is adjacent to exactly one vertex in D. By the definitions of Cayley graph and efficient dominating set, it is easy to see that if D is an efficient dominating set of $Cay(G,S)$, then

$$g(Cay(G,S)) =| D |= \frac{n}{|S|+1}. \quad (1)$$

Many authors investigated the efficient dominating sets in some families of graphs. Biggs [1] and Kratochvíl [2] first investigated the existence of efficient dominating sets in some graphs. Later Livingston and Stout [3] studied the existence and construction of efficient dominating sets in families of graphs arising from the interconnection networks of parallel computers. In the past few years efficient dominating sets in Cayley graphs have received much attention. For example, Lee [4] proved that a Cayley graph on an abelian group admits an efficient dominating set if and only if it is a covering graph of a complete graph. Dejter and Serra [5] gave a constructing tool to produce infinite families of E-chains of Cayley graphs on symmetric groups, which include the pancake graphs and the star graphs, where an E-chain is a countable family of nested graphs each containing an efficient dominating set. In addition, efficient dominating sets in circulant graphs (that is, Cayley graphs on cyclic groups) were studied by many authors. Tamizh Chelvam and Mutharasu [6] gave a necessary and sufficient condition for a subgroup to be an efficient dominating set in circulant graphs. Kumar and MacGillivray [7] characterized the efficient dominating sets in circulant graphs with domination number two and three, and Deng [8] further characterized the existence and construction of efficient dominating sets with domination number prime. Obradović et al. [9] gave necessary and sufficient conditions for the existence of efficient dominating sets in connected circulant graphs of degree 3 and 4. Feng et al. [10] and Deng et al. [11] independently generalized the results in [9] by considering the case of more general degree.

Let S_n be the symmetric group of degree n. The pancake graph P_n is the Cayley graph $Cay(S_n, PR_n)$, where $PR_n=\{r_{1j}: 2\le j\le n\}$ and $r_{1j}=\begin{pmatrix} 1 & 2 & \ldots & j & j+1 & \ldots & n \\ j & j-1 & \ldots & 1 & j+1 & \ldots & n \end{pmatrix}$.

The pancake graph is well known because of the famous open pancake problem about computing its diameter. The star graph ST_n is the Cayley graph $Cay(S_n, T_n)$, where $T_n=\{(1\ i): 2\le i\le n\}$. The star and pancake graphs are widely used in computer science as models for interconnection networks of parallel computers [12,13]. It is also known that the efficient dominating sets are used in broadcasting algorithms for multiple messages on the star and pancake graphs [14].

As a generalization of the star and pancake graphs, we here define a new class of Cayley graphs on S_n. For any given positive integers n and k with $n>k$, let $T^{(n,k)}=\{s_i : i\in\{1,2,\ldots,n\}\setminus\{k\}\}$ be an inverse-closed generating set of S_n, where each $s_i\in T^{(n,k)}$ satisfies that $k^{s_i}=i, i^{s_i}=k$, and each $s_i\in T^{(n,k)}\setminus\{s_n\}$ satisfies that $n^{s_i}=n$. Defining $\Gamma^{(n,k)}:=Cay(S_n,T^{(n,k)})$. Clearly, $\Gamma^{(n,k)}$ is connected and n-regular. By the definition of $\Gamma^{(n,k)}$, it is easy to see that the pancake graph P_n and the star graph ST_n both belong to $\Gamma^{(n,1)}$.

In [15], Konstantinova characterized all the efficient dominating sets in the star and pancake graphs. In this paper, we give some results about efficient dominating sets in $\Gamma^{(n,k)}$, which implies that the dominating number of $\Gamma^{(n,k)}$ is $(n-1)!$.

2. Main Result

2.1. Preliminary lemmas

Let G be a finite group with multiplicative notation. We define the product of two subsets M,N of G by $MN=\{mn: m\in M, n\in N\}$. If each $x\in MN$ has a unique representation in the form $x=mn$ with $m\in M$ and $n\in N$, then the product MN is called direct, denoted by $M\times N$.

Lemma 2.1 [16] *Let* G be a finite group and let M,N be subsets of G. Then $G=M\times N$ if and only if $M^{-1}M\cap NN^{-1}=\{e\}$ and $|G|=|M||N|$.

By the definitions of Cayley graph and efficient dominating set, one can easily obtain the following lemma.

Lemma 2.2 Let S be a subset of a finite group G not containing the identity e, and let $S_e=S\cup\{e\}$. Then $Cay(G,S)$ admits an efficient dominating set D if and only if $G=S_e\times D$.

2.2. The domination number

Theorem 2.3 Let $B^i=\{p\in S_n : k^p=i\}$ for $i=1,2,\ldots,n$. Then B^i $(i=1,2,\ldots,n)$ are efficient dominating sets in $\Gamma^{(n,k)}$. In particular, the domination number of $\Gamma^{(n,k)}$ is $(n-1)!$.

Proof. Let $T_e^{(n,k)}=T^{(n,k)}\cup\{e\}$. For any distinct $p_1,p_2\in B^i$, clearly $k^{p_1p_2^{-1}}=i^{p_2^{-1}}=k$. On the other hand, for any distinct $s_i,s_j\in T^{(n,k)}$, clearly $k^{s_i^{-1}s_j}=i^{s_j}\neq k$. So $(T^{(n,k)})^{-1}T^{(n,k)}\cap B_iB_i^{-1}=\{e\}$. Similarly, for any element s of $T^{(n,k)}$ or $(T^{(n,k)})^{-1}$, clearly $k^s\neq k$. Thus $T^{(n,k)}\cap B_iB_i^{-1}=\varnothing$, $(T^{(n,k)})^{-1}\cap B_iB_i^{-1}=\varnothing$. Set $T_e^{(n,k)}=T^{(n,k)}\cup\{e\}$. Then $(T_e^{(n,k)})^{-1}T_e^{(n,k)}\cap B_iB_i^{-1}=\{e\}$. Since $|S_n|=|T_e^{(n,k)}||B^i|$, it follows from Lemma 2.1 that $S_n=T_e^{(n,k)}\times B^i$. By Lemma 2.2, B^i $(i=1,2,\ldots,n)$ are efficient dominating sets in $\Gamma^{(n,k)}$, which implies that the domination number of $\Gamma^{(n,k)}$ is $|B_i|=(n-1)!$. The assertion holds.

Theorem 2.4 Let $B^i=\{p\in S_n : k^p=i\}, B_j=\{p\in S_n : n^p=j\}$, $B_j^i=B^i\cap B_j$ for any $i,j\in\{1,2,\ldots,n\}$. Then the following (i)-(ii) hold:

(i) For any distinct $i,j\in\{1,2,\ldots,n\}$, each vertex in B^i is adjacent to exactly one vertex in B^j;

(ii) For any distinct $i,j\in\{1,2,\ldots,n\}$, each vertex in B_j^i is adjacent to exactly one vertex in B_i^j and exactly one vertex in B_j^l for each $l\neq i,j$.

Proof. (i) By Theorem 2.3, B^i and B^j are both efficient dominating sets in $\Gamma^{(n,k)}$. By the definition of efficient dominating set, it is easy to see that each vertex in B^i is adjacent to exactly one vertex in B^j. Thus the assertion holds.

(ii) For any $p\in B_j^i$, clearly $k^p=i$ and $n^p=j$. Since for $s_n\in T^{(n,k)}$, we have $k^{s_np}=n^p=j$ and $n^{s_np}=k^p=i$, it follows that $s_np\in B_i^j$. Since for any $s_m\in T^{(n,k)}\setminus\{s_n\}$, $k^{s_mp}=m^p$ and $n^{s_mp}=n^p=j$, it follows that $s_mp\in B_j^{m^p}$. Next we claim that $m^p\neq i,j$. Otherwise, if $m^p=i$, then $m=k$, which contradicts with $s_m\in T^{(n,k)}\setminus\{s_n\}$. Similarly, if $m^p=j$, then $m=n$, which also contradicts with $s_m\in T^{(n,k)}\setminus\{s_n\}$. Set $l=m^p$. Then each vertex in B_j^i is adjacent to exactly one vertex in B_i^j and exactly one vertex in B_j^l for each $l\neq i,j$. The assertion holds.

2.3. Efficient dominating sets

Theorem 2.5 Let D be any efficient dominating set in $\Gamma^{(n,k)}$ and let $D_j=D\cap B_j$ and $D_i^j=D\cap B_i^j$ for any $i,j\in\{1,2,\ldots,n\}$. Then $|D_j|+|D_i^j|=(n-2)!$ for any distinct $i,j\in\{1,2,\ldots,n\}$.

Proof. By Theorem 2.4, each vertex in B_j^i is adjacent to exactly one vertex in B_i^j and exactly one vertex in B_j^l

for each $l \neq i, j$, which implies that $N(B_j^i) = (\bigcup_{l \neq i,j} B_j^l) \cup B_i^j$.

By the definition of efficient dominating set, each vertex in B_j^i either belongs to D_j^i or is adjacent to exactly one vertex in $(\bigcup_{l \neq i,j} D_j^l) \cup D_i^j$. Note that $D_j = \bigcup_{l \neq j} D_j^l$. Hence

$$(n-2)! = |B_j^i| = |D_j^i| + |\bigcup_{l \neq i,j} D_j^l| + |D_i^j| = |D_j| + |D_i^j|,$$

the assertion holds.

Theorem 2.6 Let D be any efficient dominating set in $\Gamma^{(n,k)}$ and let $D^j = D \cap B^j$ for any $j \in \{1, 2, \ldots, n\}$. Then the following (i)-(ii) hold:

(i) $|D_{i_1}^j| = |D_{i_2}^j|$ for any distinct $i_1, i_2 \in \{1, 2, \ldots, n\} \setminus \{j\}$.

(ii) $|D^j| + (n-1)|D_j| = (n-1)!$ for any $j \in \{1, 2, \ldots, n\}$. In particular, $|D^j|$ is divided by $n-1$ for any $j \in \{1, 2, \ldots, n\}$.

Proof. (i) By Theorem 2.5, $|D_i^j| = (n-2)! - |D_j|$ for any $i \in \{1, 2, \ldots, n\} \setminus \{j\}$, and thus $|D_{i_1}^j| = |D_{i_2}^j|$.

(ii) Since $D^j = \bigcup_{i \neq j} D_i^j$, it follows from (i) that $|D^j| = (n-1)|D_i^j|$, which together with Theorem 2.5 implies that $|D^j| = (n-1)((n-2)! - |D_j|)$. Thus, $|D^j| + (n-1)|D_j| = (n-1)!$ and $|D^j|$ is divided by $n-1$. Hence the assertion holds.

Theorem 2.7 Let D be any efficient dominating set in $\Gamma^{(n,k)}$. Then $D_j = \varnothing$ for some $j \in \{1, 2, \ldots, n\}$ if and only if $D = B^j$.

Proof. By Theorem 2.6 (ii), $|D^j| + (n-1)|D_j| = (n-1)!$. If $D_j = \varnothing$, then $|D^j| = (n-1)!$, which together with $D^j \subseteq B^j$ and $|B^j| = (n-1)!$ implies that $D^j = B^j$. Moreover, since $D_j \subseteq D$ and $|D| = (n-1)!$, it follows that $D^j = D$. Therefore, $D = B^j$. If $D = B^j$, then $D^j = B^j$, and thus $|D^j| = |B^j| = (n-1)!$, which together with Theorem 2.6 (ii) implies that $D_j = \varnothing$. The assertion holds.

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References

[1] N. Biggs, Perfect codes in graphs, J. Combin. Theory Ser. B 15 (1973) 288-296.
[2] J. Kratochvíl, Perfect codes of graphs, J. Combin. Theory Ser. B 40 (1986) 224-228.
[3] M. Livingston and Q. F. Stout, Perfect dominating sets, Congressus Numerantium 79 (1990) 187-203.
[4] J. Lee, Independent perfect domination sets in Cayley graphs, J. Graph Theory 37 (2001) 213-219.
[5] I. J. Dejter and O. Serra, Efficient dominating sets in Cayley graphs, Discrete Applied Math. 129(2003) 319-328.
[6] T. Tamizh Chelvam and S. Mutharasu, Subgroups as efficient dominating sets in Cayley graphs, Discrete Appl. Math. 161 (2013)1187-1190.
[7] K. R. Kumar and G. MacGillivray, Efficient domination in circulant graphs, Discrete Math. 313 (2013) 767-771.
[8] Y. P. Deng, Efficient dominating sets in circulant graphs with domination number prime, Inform. Process. Lett. 114 (2014) 700-702.
[9] N. Obradović, J. Peters and G.~Ružić, Efficient domination in circulant graph with two chord lengths, Inform. Process. Lett. 102 (2007) 253-258.
[10] R. Q. Feng, H. Huang and S. M. Zhou, Perfect codes in circulant graphs, Discrete Math. 340 (2017) 1522-1527.
[11] Y. P. Deng, Y. Q. Sun, Q. Liu and H. C. Wang, Efficient dominating sets in circulant graphs, Discrete Math. 340 (2017) 1503-1507
[12] S. B. Akers and B. Krishnamurthy, A group-theoretic model for symmetric interconnection networks, IEEE Trans. Comput. (4)38(1989) 555-566.
[13] S. Lakshmivarahan, J. S. Jwo, S. K. Dhall, Symmetry in interconnection networks based on Cayley graphs of permutation groups: A survey, Parallel Comput. (4)19(1993) 361-407.
[14] K. Qiu, Optimal broadcasting algorithms for multiple messages on the star and Pancake graphs using minimum dominating sets, Congressus Numerantium 181(2006) 33-39.
[15] E. Konstantinova, On Some Structural Properties of Star and Pancake Graphs, LNCS, 7777(2013) 472-487.
[16] Y. P. Deng, Automorphism groups of a family of Cayley graphs of the alternating groups, Ars Combin. 121 (2015) 65-70