The Inequality Proofs and Applications of Function's Strict Convexity

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Abstract: Convexity is an important property of functions, which has many applications in geometry, function optimization, economics, and other problems. In this paper, we mainly study and prove some theorems and properties about strictly convex functions and use examples to illustrate the importance of these results. The proofs of these theorems provide a theoretical basis of solving problems about strict inequalities of the strictly convex functions.

Keywords: Strictly convex function; Strict Jensen's inequality; Strictly convex application

1. Introduction

Convex function is the main content of convex analysis, which is widely used in mathematical economy, engineering, management science and optimization theory, especially in the branches of mathematical programming [1]. Intuitively, it is easy to compare the increase rate between functions on defined intervals with the property that the derivative of the function does not decrease. It is important to note that the properties of functions are usually considered on certain intervals, so only if we know the given interval can we use the related properties. In semi-strictly convex functions are studied, and the additivity theory of upper convex function and lower convex function is introduced, as well as the properties of integral. In this paper, referring to [2], upper convex function is called convex function, and lower convex function is called concave function. [3] mainly proves the properties and discuss applications of strictly convex functions from the perspective of set through convex sets (strict Jensen's inequality). In this paper, the equivalent proof of strictly convex functions is easy to understand, and there are several relevant examples. We should mention that there are similar results in related research, but the equivalence proof of strictly convex function is not given [4]. In the following, we give the definition of strictly convex function, some useful properties and their proof, and the application of these properties in the form of examples.

2. Proof of the Strict Inequality of a Lemma

Firstly, we give the definition of strictly convex (concave) function as follows.

Definition 1[5] f is defined as a function on the interval *I*. If for any two points x_1, x_2 on *I*, and any real number $\lambda \in$ （0, 1） the inequality $f(\lambda x_1+(1-\lambda)x_2)<\lambda f(x_1)+(1-\lambda)f(x_2)$ holds, then f is called a strictly convex function; otherwise, if $f(\lambda x_1+(1-\lambda)x_2)>\lambda f(x_1)+(1-\lambda)f(x_2)$ holds, then f is called a strictly concave function.

Lemma 1. The necessary and sufficient condition for f to be strictly convex function on I is: For any three points on the I . $x_1<x_2《<x_3$, the following inequality (1) always holds.

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}<\frac{f(x_3)-f(x_2)}{x_3-x_2} \tag{1}$$

The proof of necessity

Denote $\lambda=\frac{x_3-x_2}{x_3-x_1}$, $x_2=\lambda x_1+(1-\lambda)x_3$, $(0<\lambda<1)$. Since f is a strictly convex function, according to definition 1, we have

$$f(x_2)=f(\lambda x_1+(1-\lambda)x_3)<\lambda f(x_1)+(1-\lambda)f(x_3)=\frac{x_3-x_2}{x_3-x_1}f(x_1)+\frac{x_2-x_1}{x_3-x_1}$$

, and then the following inequalities holds:

$((x_3-x_1)f(x_2)<(x_3-x_2)f(x_1)+(x_2-x_1)f(x_3)$;

$((x_3-x_2)f(x_2)+(x_2-x_1)f(x_2)<(x_3-x_2)f(x_1)+(x_2-x_1)f(x_3)$
(2)

After transposing the terms of formula (2), it is easy to obtain the inequality (1).

The proof of sufficiency

Take two points in the interval I . $x_1, x_3 (x_1<x_3)$, and take one point in the interval $[x_1, x_3]$, then $x_2=\lambda x_1+(1-\lambda)x_3, \lambda\in(0,1)$, $\lambda=\frac{x_3-x_2}{x_3-x_1}$.

Since $\frac{f(x_2)-f(x_1)}{x_2-x_1}<\frac{f(x_3)-f(x_2)}{x_3-x_2}$, we have the following inequalities:

$$((x_3-x_2)(f(x_2)-f(x_1))<(f(x_3)-f(x_2))(x_2-x_1)$$
$$(x_3-x_2)f(x_2)+(x_2-x_1)f(x_2)<(x_3-x_1)f(x_3)$$.

Then

$f(x_2)<\frac{x_3-x_2}{x_3-x_1}f(x_1)+\frac{x_2-x_1}{x_3-x_1}f(x_3)$ and $f(\lambda x_1+(1-\lambda)x_3)<\lambda f(x_1)+(1-\lambda)f(x_3)$ holds. Therefore, f is a strictly concave function on I. This finished the proof.

Example 1. Set f as a function defined on $[1,+\infty)$. It satisfies $f(x)>0, f^2(1)-f(1)=0. \forall x_1,x_2\in[1,+\infty)$, and $1《<x_1<x_2,(x_2-1)f(x_1)+x_1<((x_1-1)f(x_2)+x_2$.

Please prove that f is the strictly concave function defined on $[1,+\infty)$.

Proof: Because $f^2(1)-f(1)=0, f(x)>0, \therefore f(1)=1$, and

$$(x_2-1)f(x_1)+x_1<(x_1-1)f(x_2)+x_2;$$
$$x_2 f(x_1)-f(x_1)+x_1<x_1 f(x_2)-f(x_2)+x_2;$$
$$(x_2-x_1)(f(x_1)-1)<(x_1-1)(f(x_2)-f(x_1));$$
$$\frac{f(x_1)-1}{x_1-1}<\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

Therefore f is the strictly concave function defined on $[1,+\infty)$.

3. Equivalent Conditions of Strict Convexity of Derivable Function

Theorem 1. If f is a differentiable function on interval I, the following statements are equivalent to each other:

f is the strictly concave function defined on I.

f' is a strict function on I.

Any two points, x_1,x_2, on I. $f(x_2)>f(x_1)+f'(x_1)(x_2-x_1)$.

Proof: (1⇒2) Take any three points on the interval I, $x_1,x_2,x_3(x_1<x_2<x_3)$ and a sufficiently small positive h. Because $x_1-h<x_1<x_2<x_3<x_3+h$, according to the property of strict convexity and Lemma 1, we have

$$\frac{f(x_1)-f(x_1-h)}{h}<\frac{f(x_2)-f(x_1)}{x_2-x_1}<\frac{f(x_3)-f(x_2)}{x_3-x_2}<\frac{f(x_3+h)-f(x_3)}{h}$$

Since f is a differentiable function, when $h\to 0^+$, it is easy to obtain

$$f'(x_1)\le\frac{f(x_2)-f(x_1)}{x_2-x_1}<\frac{f(x_3)-f(x_2)}{x_3-x_2}\le f'(x_3)\Rightarrow f'(x_1)<f'(x_3)$$

For the arbitrariness of x1 and x3, f' is a strict function on I. (2⇒3) On interval $[x_1,x_2](x_1<x_2)$, apply the Lagrange mean value theorem, and since f' is strictly increasing function, the following inequalities holds:

$$f(x_2)-f(x_1)=f'(\xi)(x_2-x_1)>f'(x_1)(x_2-x_1),$$

$f(x_2)>f(x_1)+f'(x_1)(x_2-x_1)$. (3⇒1) Given any two points x_1,x_2 on I, $x_3=\lambda x_1+(1-\lambda)x_2,(0<\lambda)$ According to 3). $x_1-x_3=(1-\lambda)(x_1-x_2)$ and $x_2-x_3=\lambda(x_2-x_1)$, we have

$$f(x_1)>f(x_3)+f'(x_3)(x_1-x_3)=f(x_3)+(1-\lambda)f'(x_3)(x_1-x_2);$$
$$f(x_2)>f(x_3)+f'(x_3)(x_2-x_3)=f(x_3)+\lambda f'(x_3)(x_2-x_1).$$

And then the following inequality can be obtained:

$$\lambda f(x_1)+(1-\lambda)f(x_2)>f(x_3)=f(\lambda x_1+(1-\lambda)x_2).$$

Therefore f is the strictly concave function defined on I. Theorem 1 is proved.

Example 2. $f(x)=x\ln x(x>0)$, prove that

$f(x)$ is the strictly concave function defined on $(0,+\infty)$. $\forall x>1, e^{(\frac{1}{x}-1)}>\frac{1}{x}$.

Proof: (1) $f'(x)=1+\ln x, f''(x)=\frac{1}{x}>0$

So $f(x)$ is the strict function on $(0,+\infty)$, According to the reciprocity theorem above, we can see $f(x)$ is the strictly concave function defined on $(0,+\infty)$.

$\forall x>1$, because $(2\to 3)$, get $f(x)>f(1)+f'(1)(x-1)$

$x\ln x>1-\frac{1}{x}, \ln x+\frac{1}{x}>1$. Because e^x is an increasing function on R, $e^{\ln x+\frac{1}{x}}>e, xe^{\frac{1}{x}}>e, e^{(\frac{1}{x}-1)}>\frac{1}{x}$.

4. The Necessary and Sufficient Condition of Convexity of Two-order Derivable Functions

Theorem 2. Let f be a two-order derivable function on interval I, then f is a strictly concave function if and only if $f''(x)\ge 0, x\in I$, and $\forall(a,b)\subseteq I, f''(x)$ is not always identical to zero.

Proof: (1) "⇒": According to theorem 1, it can be seen that if f is a two-order derivable and strictly concave function, and then $f'(x)$ is a strictly increasing function on interval I. This means that

$(I)\forall x\in I, f''(x)\geq 0$; and(II) $\forall(a,b)\subseteq I, f''(x)$ is not identical to zero.

(2) "$\Leftarrow$": if $f''(x)\geq 0\quad \forall x\in I$,and $\forall(a,b)\subseteq I, f''(x)$ is not identical to zero, then it is easy to know that $f'(x)$ is strictly increasing on interval I. According to theorem 1, we can get the result that $f(x)$ is a strictly concave function on interval I. This completes the proof of theorem 2.

Example 4. $f(x)=e^x-\frac{1}{2}x^2, x\geq 0$. Prove $f(x)$ is strictly concave function.

Proof: $f(x)=e^x-\frac{1}{2}x^2, f''(x)=e^x-1$, and $e^x-1\geq 0, x\geq 0$.

But only when $x=0$, $f''(x)=0$. That is, $f(x)$ does not always equal to zero on $[0,+\infty)$. According to theorem 2, $f(x)$ is the strictly concave function.

5. Strict Proof of Jensen's Inequality

Theorem 3. If f is the strictly concave function on interval $[a,b]$, there are at least two unequal points, $x_1,x_2(x_1\neq x_2)$, $\forall\ x_i\in[a,b],\lambda_i>0(i=1,2,\cdots,n),\sum_{i=1}^{n}\lambda_i=1$, (3)

Proof: We apply mathematical induction. When n=2, according to definition 1 the proposition is clearly true. Suppose when n=k the proposition is true. $\forall$ $x_1,x_2\cdots,x_k\in[a,b]$ and there are at least two unequal points. set $x_1,x_2(x_1\neq x_2)$ and $\alpha_i>0, i=1,2,\cdots,k,\sum_{i=1}^{n}\alpha_i=1,$

We get $f(\sum_{i=1}^{k}\alpha_i x_i)<\sum_{i=1}^{k}\alpha_i f(x_i).$

Since $x_1,x_2,\cdots x_k,x_{k+1}\in[a,b]$,

$\lambda_i>0(i=1,2,\cdots,k+1)),\sum_{i=1}^{k+1}\lambda_i=1.$

Denote $\alpha_i=\frac{\lambda_i}{1-\lambda_{k+1}}, i=1,2,\cdots,k,\quad \sum_{i=1}^{k}\alpha_i=1.$

Through mathematical induction.

$$
\begin{aligned}
&f(\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_kx_k+\lambda_{k+1}x_{k+1})\\
&=f((1-\lambda_{k+1})\frac{\lambda_1x_1+\lambda_2x_2+\cdots\lambda_kx_k}{1-\lambda_{k+1}}+\lambda_{k+1}x_{k+1})\\
&<((1-\lambda_{k+1})f(\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_kx_k)+\lambda_{k+1}f(x_{k+1})\\
&<(1-\lambda_{k+1})[\alpha_1f(x_1)+\alpha_2f(x_2)+\cdots+\alpha_kf(x_k)]+\lambda_{k+1}f(x_{k+1})\\
&=(1-\lambda_{k+1})\left[\frac{\lambda_1}{1-\lambda_{k+1}}f(x_1)+\frac{\lambda_2}{1-\lambda_{k+1}}f(x_2)+\cdots+\frac{\lambda_k}{1-\lambda_{k+1}}f(x_k)\right]\\
&+\lambda_{k+1}f(x_{k+1})=\sum_{i=1}^{k+1}f(x_i).
\end{aligned}
$$

$$f(\sum_{i=1}^{n}\lambda_ix_i)<\sum_{i=1}^{n}\lambda_if(x_i)$$

This proves that for any positive integer n (≥ 2), if f is the strictly concave functions, (3)establishment.

Example 5. Given $a_i>0,(i=1,2,\cdots,n)$, and there are at least two unequal points, say $a_1,a_2(a_1\neq a_2)$. Proof inequality $\sqrt[n]{a_1a_2\cdots a_n}<\frac{a_1+a_2+\cdots+a_n}{n}$

Proof: Assume $f(x)=-\ln x, f'(x)=-\frac{1}{x}, f''(x)=\frac{1}{x^2}$, then when $x>0$, $f(x)$ is k increasing functions. According to theorem 3, we have

$$
\begin{aligned}
&-\ln(\frac{a_1+a_2+\cdots+a_n}{n})<-\frac{1}{n}(\ln a_1+\ln a_2+\cdots\ln a_n);\\
&\ln(a_1a_2\cdots a_n)^{\frac{1}{n}}<\ln(\frac{a_1+a_2+\cdots a_n}{n});\\
&\sqrt[n]{a_1a_2\cdots a_n}<\frac{a_1+a_2+\cdots+a_n}{n}.
\end{aligned}
$$

Then this problem is proved.

6. Conclusions

In this paper, the definition of convex (concave) function in mathematical analysis book is extended to the definition of strictly convex (concave) function, and the related theorems of convex function (mainly non-strict inequalities) are extended to strict inequalities. Once we have completed the proofs in this paper, we can directly use the strict inequalities to solve related problems which can save a lot of tedious proofs each time.

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