Application of Matrix Diagonalization in Calculation

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Abstract: As a special kind of matrix, diagonalizable matrix is widely used in higher order of square matrix, differential equation, matrix function, vector space, linear transformation, etc.

Keywords: Matrix diagonalization; Characteristic value; Invertible matrix

1. Introduction

In linear algebra, matrix diagonalization occupies a very important position, its calculation, to solve the homogeneous linear equations with determinant, matrix inverse, linear transformation, quadratic tightly linked, etc. But as a special kind of diagonalization matrix, the phalanx of high order, the differential equation and matrix function, vector space, linear transformation, etc, has a wide range of applications.

2.The Definition of Diagonalizable Matrix

Definition 1: For n-rank square matrix of order A,B. If there is an invertible matrix P, which satisfies.

$$P^{-1}AP = B \quad (1)$$

We may call the matrix A is similar with B, and P is the similar transformation matrix that turns A into B . If B is A diagonal matrix, we say A is diagonalizable.

3.Conditions and Methods of Matrix Diagonalization

Theorem 1 If the n order matrix has n different eigenvalues, then it can be diagonalized.

Theorem 2 If any of the k -weight eigenvalues λ for the n - matrix A satisfies $r(A-\lambda E) = n-k$. Then A can be diagonalized.

Basic steps of matrix diagonalization:

(1) use $|A-\lambda E| = 0$ to obtain all the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$;

(2) calculate the homogeneous linear equations $(A-\lambda_i E)x = 0$ for each k_i -weight eigenvalue λ_i , so as to obtain n linearly independent eigenvector; $p_1, p_2, \cdots, p_n$;

(3) let $P = (p_1, p_2, \cdots, p_n)$,then $P^{-1}AP = diag(\lambda_1, \lambda_2, \cdots \lambda_n)$.

4. The Application of Matrix Diagonalization

4.1.The application for the power of square matrix

If A can be diagonalized, that is, there is an invertible matrix P to make $P^{-1}AP = \Lambda$,
then $A = P\Lambda P^{-1}$,so $A^n = P\Lambda^n P^{-1}$.

Example 1 (population mobility problem):

Given that one tenth of the urban population flows to the countryside and two tenth of the rural population flows to the cities each year, given that the total population remains unchanged, will the national population be concentrated in the cities after many years?

The original urban and rural population was set as, and the urban and rural population at the end of the first year was y_0, z_0 , then

$$\begin{cases} y_1 = 0.9y_0 + 0.2z_0 \\ z_1 = 0.1y_0 + 0.8z_0 \end{cases} \quad (2)$$

namely

$$\begin{pmatrix} y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix} \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \quad (3)$$

So the urban and rural population at the end of the k year is

$$\begin{pmatrix} y_k \\ z_k \end{pmatrix} = \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix}^k \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \quad (4)$$

Let $A = \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix}$, we can calculate A^k using matrix diagonalization,

$$\text{Then } \begin{pmatrix} y_k \\ z_k \end{pmatrix} = (y_0 + z_0) \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} + (y_0 - 2z_0) 0.7^k \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix} \quad (5)$$

Therefore, at that time, urban population and rural population is 2:1, which tends to be stable when $k \to \infty$.

4.2.The application of matrix diagonalization in the square root of matrix

Example 2

Assumes that the n square matrix A of order can be diagonalized, n order matrix B, so $B^2 = A$.

Because A can be diagonalized, there exists an invertible matrix P, so that

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \quad (6)$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are n eigenvalues of A, then

$$A = P\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}P^{-1} \quad (7)$$

$$= \left[P\begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n} \end{pmatrix}P^{-1}\right]\left[P\begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n} \end{pmatrix}P^{-1}\right] \quad (8)$$

That is

$$B = P\begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n} \end{pmatrix}P^{-1} \quad (9)$$

Clearly, B is not unique.

$$\text{For example: given } A = \begin{pmatrix} 5 & -3 & 2 \\ 6 & -4 & 4 \\ 4 & -4 & 5 \end{pmatrix}, \quad (10)$$

find B.

It is easy to know that there is an invertible matrix

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} \quad (11)$$

So

$$P^{-1}AP = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix} \quad (12)$$

then

$$A = P\begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}P^{-1} = \left[P\begin{pmatrix} 1 & & \\ & \sqrt{2} & \\ & & \sqrt{3} \end{pmatrix}P^{-1}\right]\left[P\begin{pmatrix} 1 & & \\ & \sqrt{2} & \\ & & \sqrt{3} \end{pmatrix}P^{-1}\right] \quad (13)$$

So

$$B = P\begin{pmatrix} 1 & & \\ & \sqrt{2} & \\ & & \sqrt{3} \end{pmatrix}P^{-1} = \begin{pmatrix} -2+2\sqrt{2}+\sqrt{3} & 2-\sqrt{2}-\sqrt{3} & -1+\sqrt{3} \\ -4+2\sqrt{2}+2\sqrt{3} & 4-\sqrt{2}-2\sqrt{3} & -2+2\sqrt{3} \\ -2+2\sqrt{3} & 2-2\sqrt{3} & -1+2\sqrt{3} \end{pmatrix} \quad (14)$$

4.3. The application of matrix diagonalization in matrix functions

Example 3

$$\text{If} \quad A = \begin{pmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{pmatrix} \quad (15)$$

what is e^{At} and $\cos A$?

Solution: If A can be diagonalized, then there is an invertible matrix P that

$$\text{makes } A^n = P\Lambda^n P^{-1} \quad (16)$$

and

$$e^{At} = E + At + \frac{1}{2!}(At)^2 + \cdots + \frac{1}{n!}(At)^n + \cdots \quad (17)$$

then

$$e^{At} = PEP^{-1} + P(\Lambda t)P^{-1} + \frac{1}{2!}P(\Lambda t)^2P^{-1} + \cdots + \frac{1}{n!}P(\Lambda t)^n P^{-1} + \cdots$$

$$= P[E + (\Lambda t) + \frac{1}{2!}(\Lambda t)^2 + \cdots + \frac{1}{n!}(\Lambda t)^n + \cdots]P^{-1}$$

$$= P\begin{pmatrix} e^{-\lambda_1 t} & & \\ & e^{-\lambda_2 t} & \\ & & e^{-\lambda_3 t} \end{pmatrix}P^{-1} \quad (18)$$

In this case, there is an invertible mtrix

$$P = \begin{pmatrix} -1 & -2 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad (19)$$

So

$$P^{-1}AP = \begin{pmatrix} -2 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad (20)$$

then

$$e^{At} = P\begin{pmatrix} e^{-2t} & & \\ & e^{t} & \\ & & e^{t} \end{pmatrix}P^{-1} = \begin{pmatrix} 2e^{t}-e^{-2t} & 2e^{t}-2e^{-2t} & 0 \\ e^{-2t}-e^{t} & 2e^{-2t}-e^{t} & 0 \\ e^{-2t}-e^{t} & 2e^{-2t}-2e^{t} & e^{t} \end{pmatrix} \quad (21)$$

In the same way

$$\begin{pmatrix} \cos(-2) & & \\ & \cos 1 & \\ & & \cos 1 \end{pmatrix} P^{-1} = \begin{pmatrix} 2\cos 1-\cos 2 & 2\cos 1-2\cos 2 & 0 \\ \cos 2-\cos 1 & 2\cos 2-\cos 1 & 0 \\ \cos 2-\cos 1 & 2\cos 2-2\cos 1 & \cos 1 \end{pmatrix} \tag{22}$$

4.4. The application in solving ordinary differential equations

Example 4 Solve linear differential equations with constant coefficients

$$\begin{cases} \dot{x}_1 = 4x_1 + x_3 \\ \dot{x}_2 = 2x_1 + 3x_2 + 2x_3 \\ \dot{x}_3 = x_1 + 4x_3 \end{cases} \tag{23}$$

Where

$$x_i = x_i(t), \dot{x}_i = \frac{dx_i(t)}{dt} \tag{24}$$

The solution of this differential equation can be expressed as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{25}$$

It's easy to know that there's a matrix

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \tag{26}$$

that makes

$$P^{-1} \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix} P = \begin{pmatrix} 5 & & \\ & 3 & \\ & & 3 \end{pmatrix} \tag{27}$$

then

$$\begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix} = P \begin{pmatrix} 5 & & \\ & 3 & \\ & & 3 \end{pmatrix} P^{-1} \tag{28}$$

plug in the original system

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 5 & & \\ & 3 & \\ & & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{29}$$

Let

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{30}$$

Then the above system is

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} 5 & & \\ & 3 & \\ & & 3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \tag{31}$$

namely

$$\dot{z}_1 = 5z_1, \ \dot{z}_2 = 5z_2, \dot{z}_3 = 5z_3$$

We can obtain

$$z_1 = c_1 e^{5t}, \ z_2 = c_2 e^{3t}, z_3 = c_3 e^{3t}$$

Which means

$$\begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{pmatrix} = \begin{pmatrix} e^{5t} & & \\ & e^{3t} & \\ & & e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \tag{32}$$

so

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{pmatrix} \tag{33}$$

$$= \begin{pmatrix} c_1 e^{5t} + c_2 e^{3t} \\ 2c_1 e^{5t} + c_3 e^{3t} \\ c_1 e^{5t} - c_2 e^{3t} \end{pmatrix} \tag{34}$$

In addition, matrix diagonalization is widely used in determinant calculation, vector space,linear transformation and image processing.

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