An Improved Inexact Newton Method for Nonlinear Equations

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Abstract: In this paper, we describe a variant of the Inexact Newton method for solving the nonlinear equations. We make a study of a new nonmonotone inexact Newton method with a nonmonotone backtracking strategy. To decrease the computational complexity, the BFGS update formula is used to generate an approximated matrix rather than a normal Jacobian matrix. Theoretical analysis indicates that the new method preserves the global convergence under mild conditions.

Keywords: Nonlinear equations; Inexact Newton method; Nonmonotone strategy; Global convergence

1. Introduction

Consider the following nonlinear system of equations:

$$F(x)=0,\ x\in R^n. \tag{1}$$

where $F:R^n\to R^n$ is continuously differentiable. There are various methods to solve the problem (1), such as the Newton and the quasi-Newton methods [1-6], the spectral method [7, 8], the trust-region-based methods [9-12]. Suppose that $F(x)$ has a zero, then the nonlinear system (1) is equivalent to the following nonlinear unconstrained least-squares problem

$$\min f(x):=\frac{1}{2}\|F(x)\|^2 \quad s.t.\ x\in R^n. \tag{2}$$

where $\|.\|$ denotes the Euclidean norm.

The general iterative formula for (1) proceed as follows: given a point x_k, find a descent direction d_k, a suitable step length α_k and construct the new point as follows:

$$x_{k+1}=x_k+\alpha_k d_k. \tag{3}$$

The Newton's method is a classical way for solving the nonlinear equations because it converges rapidly from any sufficiently good starting position. It has the following form to get d_k:

$$\nabla F(x_k)d_k=-F(x_k). \tag{4}$$

The main drawback of Newton's method is that the direct computation of the Jacobian is computationally expensive. This fact motivated the development of quasi-Newton method. The quasi-Newton method is of the form

$$B_k d_k=-F(x_k). \tag{5}$$

where B_k is generated by the BFGS formula

$$B_{k+1}=B_k-\frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k}+\frac{y_k y_k^T}{y_k^T s_k}. \tag{6}$$

where $s_k=x_{k+1}-x_k$ and $y_k=F_{k+1}-F_k$.

Similar with [13], we propose a new Inexact Newton method to substitute (5) with a condition on its residual:

$$B_k d_k=-F_{l(k)}+\gamma_k. \tag{7}$$

where $\frac{\|\gamma_k\|}{\|F_k\|}\le\beta_k,\ \beta_k\in\left[0,\frac{1}{2}\right)$, and

$F_{l(k)}:=\max_{0\le j\le n(k)}\left\{\|F_{k-j}\|\right\},\quad k\in N\cup\{0\},$

$n(0)=0\ and\ 0\le n(k)\le\min\{n(k-1)+1,N\}\ with\ N>0$.

As for computing a suitable α_k, after studying the methods from [6,14-16], we make a study of the new inexact quasi-Newton method with a new nonmonotone backtracking strategy for solving the nonlinear equations. Actually, at the k th iteration of our algorithm, we combine the method in [16] and the new nonmonotone technique proposed in [17] to obtain the step size α_k,

$$f(x_k+\alpha_k d_k)\le R_k+\alpha_k{}^2\sigma F(x_k)^T d_k. \tag{8}$$

where $\sigma\in(0,1/2)$ is a constant and d_k is a solution of (7).

$$R_k=\eta_k f_{l(k)}+(1\text{-}\eta_k)f_k. \tag{9}$$

where

$$f_{l(k)}=\max_{0\le j\le m(k)}\{f_{k-j}\},\quad k=0,1,2,\ldots \tag{10}$$

$m(0)=0,\ \ 0\le m(k)\le\min\{m(k-1)+1,N\},N\ge 0,$

$\eta_k\in[\eta_{\min},\eta_{\max}]\ for\ \eta_{\min}\in[0,1),\ \ \eta_{\max}\in[\eta_{\min},1].$

Because of no need to compute the Jacobian matrix $\nabla F(x)$, the storage and workload are considerably saved. Furthermore, the nonmonotone technique can improve the iterative algorithm in optimization and accelerate the convergence process.

The rest of this paper is organized as follows. In Section 2, the new algorithm will be introduced. The convergence analysis is investigated in Section 3. Finally, some conclusions are addressed in Section 4.

2. Algorithm

Now, we outline the proposed algorithm.

Algorithm 2.1

Initial: Choose a starting point $x_0 \in R^n$, an initial symmetric positive definite matrix $B_0 \in R^{n\times n}$, and constants $r \in (0,1)$, $\sigma, \beta_{\max} \in (0,1/2)$, $\varepsilon > 0$, $m(k)=0$, $n(k)=0$, let $k := 0$.

Step 1: If $\|F_k\| < \varepsilon$ holds, stop; otherwise, go to step 2.

Step2: Determine $\beta_k \in [0, \beta_{\max}]$, Solve (7) to obtain d_k.

Step3: Let $\alpha_k = 1, r, r^2, r^3, \ldots$ until (8) holds.

Step 4: Update B_k by the BFGS update formula and ensure the update matrix B_{k+1} is positive definite.

Step 5: Set $k := k+1$ and go to step 1.

Remark Step 4 of Algorithm 2.1 can ensure that B_k is always positive definite. This means that (7) has a unique solution d_k. By positive definiteness of B_k, it is easy to obtain $F_k^T d_k < 0$.

Also, we need the following standard hypothesis to complete theoretical proof.

(H1) Let the level set $\Omega = \{x \mid f(x) \le f(x_0)\}$ be bounded.

(H2) $F(x)$ is continuously differentiable on an open convex set Ω_1 containing Ω, $\{\|F_k\|\}$ is bounded.

(H3) The Jacobian of $F(x)$ is symmetric, bounded and positive definite on Ω_1, i.e., there exist positive constants $M \ge m > 0$ such that

$$\|\nabla F(x)\| \le M, \quad \forall x \in \Omega. \tag{11}$$

and

$$m\|d\|^2 \le d^T \nabla F(x) d, \quad \forall x \in \Omega, d \in R^n. \tag{12}$$

(H4) B_k is a good approximation to ∇F_k, i.e.,

$$\|(\nabla F_k - B_k) d_k\| \le \varepsilon_* \|F_k\| \tag{13}$$

where $\varepsilon_* \in \left(0, \frac{1}{2}\right)$ is a small quantity.

Considering (H4) and using the von Neumann lemma, we deduce that B_k is also bounded (see [15]).

Lemma 2.1 Let (H1)-(H2) hold and the sequence $\{x_k\}$ is generated by Algorithm 2.1, then the sequence $\{f_{l(k)}\}$ is not monotonically increasing. Therefore the sequence $\{f_{l(k)}\}$ is convergent.

Proof. Using the definition R_k and $f_{l(k)}$, we have

$$R_k = \eta_k f_{l(k)} + (1-\eta_k) f_k \le \eta_k f_{l(k)} + (1-\eta_k) f_{l(k)} = f_{l(k)} \tag{14}$$

This leads to

$$\begin{aligned} & f(x_k + \alpha_k d_k) \le \\ & R_k + \alpha_k^2 \sigma F(x_k)^T d_k \le f_{l(k)} + \alpha_k^2 \sigma F(x_k)^T d_k. \end{aligned} \tag{15}$$

The preceding inequality and the descent condition $F_k^T d_k < 0$ indicate that

$$f_{k+1} \le f_{l(k)} \tag{16}$$

On the other hand, from (10), we get

$$f_{l(k+1)} = \max_{0\le j\le m(k+1)} \{f_{k+1-j}\} \le \max_{0\le j\le m(k)+1} \{f_{k+1-j}\} = \max\{f_{l(k)}, f_{k+1}\}$$

This fact together with (16) show that the sequence $\{f_{l(k)}\}$ is not monotonically increasing. (H1) and (H2) imply that

$\exists \lambda \ s.t. \ \forall n \in N : \lambda \le f_{k+n} \le f_{l(k+n)} \le \cdots \le f_{l(k+1)} \le f_{l(k)} \le f_0$.

So $f_{l(k)}$ is convergent.

Lemma 2.2 Suppose that the sequence $\{x_k\}$ is generated by Algorithm 2.1. Then, we have

$$f_{k+1} \le R_{k+1} \quad \forall k \in N \cup \{0\}. \tag{17}$$

Proof. The proof is similar to Lemma 3.2 in [17].

3. Convergence Analysis

This section gives some convergence results under some suitable conditions.

Lemma 3.1 Suppose that (H1)-(H3) hold and the sequence $\{x_k\}$ is generated by Algorithm 2.1. Then we have

$$\lim_{k\to\infty} f_{l(k)} = \lim_{k\to\infty} f(x_k). \tag{18}$$

Proof. From (8), (10) and (14), for $k > N$, we obtain

$$\begin{aligned} f(x_{l(k)}) &= f(x_{l(k)-1} + \alpha_{l(k)-1} d_{l(k)-1}) \\ &\le R_{l(k)-1} + \sigma \alpha_{l(k)-1}^2 F_{l(k)-1}^T d_{l(k)-1} \\ &\le f(x_{l(l(k)-1)}) + \sigma \alpha_{l(k)-1}^2 F_{l(k)-1}^T d_{l(k)-1}. \end{aligned}$$

The preceding inequality together with Lemma 2.1, $\alpha_K > 0$ and $F_k^T d_k < 0$ imply that

$$\lim_{k\to\infty} \alpha_{l(k)-1}^2 F_{l(k)-1}^T d_{l(k)-1} = 0. \tag{19}$$

Based on (H1)-(H4), similar to Lemma 3.4 in [16], it is not difficult to deduce that there exist constants $M_1 \ge m_1 > 0$ such that

$$m_1\|d_k\|^2 \le d_k^T B_k d_k = -F_k^T d_k \le M_1 \|d_k\|^2. \tag{20}$$

Using (20), we have $\alpha_k^2 F_k^T d_k \le -\alpha_k^2 m_1 \|d_k\|^2$, for all k. This fact along with (19) suggest that

$$\lim_{k\to\infty} \alpha_{l(k)-1} \left\|d_{l(k)-1}\right\| = 0 \quad (21)$$

We now prove that $\lim_{k\to\infty} \alpha_k \|d_k\| = 0$. Let $\hat{l}_k = l(k+N+2)$. First, by induction, we show that, for any $j \ge 1$, we have

$$\lim_{k\to\infty} \alpha_{\hat{l}(k)-j} \left\|d_{\hat{l}(k)-j}\right\| = 0. \quad (22)$$

and

$$\lim_{k\to\infty} f\left(x_{\hat{l}(k)-j}\right) = \lim_{k\to\infty} f\left(x_{l(k)}\right). \quad (23)$$

If $j=1$, since $\{\hat{l}_k\} \subseteq \{l(k)\}$, the relation (22) directly follows from (21). The condition (22) indicates that $\left\|x_{\hat{l}(k)} - x_{\hat{l}(k)-1}\right\| \to 0$. This fact along with the fact that $f(x)$ is uniformly continuous on Ω imply that (23) holds, for $j=1$. Now, we assume that (22) and (23) hold, for a given j. Then, using (8) and (14), we obtain

$$f\left(x_{\hat{l}(k)-j}\right) \le R_{\hat{l}(k)-j-1} + \sigma \alpha_{\hat{l}(k)-j-1}^2 F_{\hat{l}(k)-j-1}^T d_{\hat{l}(k)-j-1}$$
$$\le f\left(x_{l(\hat{l}(k)-j-1)}\right) + \sigma \alpha_{\hat{l}(k)-j-1}^2 F_{\hat{l}(k)-j-1}^T d_{\hat{l}(k)-j-1}.$$

Following the same arguments employed for deriving (21), we deduce

$$\lim_{k\to\infty} \alpha_{\hat{l}(k)-(j+1)} \left\|d_{\hat{l}(k)-(j+1)}\right\| = 0$$

This means that

$$\lim_{k\to\infty} \left\|x_{\hat{l}(k)-j} - x_{\hat{l}(k)-(j+1)}\right\| = 0$$

This fact together with uniformly continuous property of $f(x)$ on Ω and (23) indicate that

$$\lim_{k\to\infty} f\left(x_{\hat{l}(k)-(j+1)}\right) = \lim_{k\to\infty} f\left(x_{\hat{l}(k)-j}\right) = \lim_{k\to\infty} f\left(x_{l(k)}\right) \quad (24)$$

Thus, we conclude that (22) and (23) hold for any $j \ge 1$.

On the other hand, for any $k \in N$, we have

$$x_{k+1} = x_{\hat{l}(k)} - \sum_{j=1}^{\hat{l}(k)-k-1} \alpha_{\hat{l}(k)-j} d_{\hat{l}(k)-j}. \quad (25)$$

From definition of $l(k)$, we have $\hat{l}(k) - k - 1 = l(k+N+2) - k - 1 \le N+1$. Thus, (22) and (25) suggest

$$\lim_{k\to\infty} \left\|x_{k+1} - x_{\hat{l}(k)}\right\| = 0. \quad (26)$$

Since $f(x)$ is uniformly continuous on Ω and (26),

$$\lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} f\left(x_{\hat{l}(k)}\right) = \lim_{k\to\infty} f\left(x_{l(k)}\right).$$

The proof is complete.

Lemma 3.2 Suppose that (H1)-(H3) hold and the sequence $\{x_k\}$ is generated by Algorithm 2.1. Then we have

$$\lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} R_k. \quad (27)$$

Proof. By Lemma 2.2 and (14), we get $f_k \le R_k \le f_{l(k)}$.

This fact together with Lemma 3.1, we have $\lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} R_k$. Then the proof is complete.

Lemma 3.3 Let (H4) hold. Then d_k is a descent direction of $f(x)$ at x_k, i.e.,

$$\nabla f(x_k)^T d_k \le -(1-\delta)\|F_k\|^2. \quad (28)$$

where $\delta \in (0,1)$.

Proof. By using (7), we get

$$\begin{aligned}\nabla f(x_k)^T d_k &= F_k^T \nabla F_k d_k \\ &= F_k^T\left[(\nabla F_k - B_k)d_k - F_{l(k)} + \gamma_k\right] \\ &\le F_k^T(\nabla F_k - B_k)d_k - \|F_k\|^2 + F_k^T\gamma_k.\end{aligned} \quad (29)$$

Thus, together with (7), we have

$$\begin{aligned}\nabla f(x_k)^T d_k + \|F_k\|^2 &\le F_k^T(\nabla F_k - B_k)d_k + F_k^T\gamma_k \\ &\le \|F_k^T\|\left[\|(\nabla F_k - B_k)d_k\| + \|\gamma_k\|\right] \\ &\le \|F_k^T\|\left[\|(\nabla F_k - B_k)d_k\| + \beta_k\|F_k\|\right]\end{aligned}$$

It follows from (13) that

$$\begin{aligned}\nabla f(x_k)^T d_k &\le \|F_k^T\|\left[\|(\nabla F_k - B_k)d_k\| + \beta_k\|F_k\|\right] - \|F_k\|^2 \\ &\le (\varepsilon_* + \beta_k - 1)\|F_k\|^2 = -(1-\delta)\|F_k\|^2.\end{aligned} \quad (30)$$

where $\delta = \varepsilon_* + \beta_k \in (0,1)$. The proof is complete.

Lemma 3.4 Let (H1)-(H4) hold. Suppose there exists a constant m_3, such that $m_3\|F_k\| \le \|d_k\|$, then Algorithm 2.1 will produce iteration $x_{k+1} = x_k + \alpha_k d_k$ in a finite number of backtracking steps.

Proof. From Lemma 3.8 in [18], we have that in a finite number of backtracking steps, α_k must satisfy

$$\|F(x_k + \alpha_k d_k)\|^2 - \|F(x_k)\|^2 \le \theta\alpha_k F(x_k)^T \nabla F(x_k) d_k.$$

By (20) and (30), we get

$$\begin{aligned}\alpha_k F(x_k)^T \nabla F(x_k) d_k &\le -\alpha_k(1-\delta)\|F_k\|^2 \\ &= -\alpha_k(1-\delta)\frac{F_k^T d_k}{F_k^T d_k}\|F_k\|^2 \\ &\le \alpha_k(1-\delta)\frac{F_k^T d_k}{m_1\|d_k\|^2}\frac{\|d_k\|^2}{m_3^2} \\ &= \alpha_k(1-\delta)\frac{F_k^T d_k}{m_1 m_3^2}.\end{aligned} \quad (31)$$

By $\alpha_k \le 1$, we have

$$\alpha_k F(x_k)^T \nabla F(x_k) d_k \le \alpha_k (1-\delta)\frac{F_k^T d_k}{m_1 m_3^2} \le \alpha_k^2 (1-\delta)\frac{F_k^T d_k}{m_1 m_3^2}. \quad (32)$$

From Lemma 2.2, we get $f_k \le R_k$. So we get

$$f(x_k+\alpha_k d_k)-R_k \le$$

$$f(x_k+\alpha_k d_k)-f_k \le \frac{1}{2}\theta\alpha_k^2(1-\delta)\frac{F_k^T d_k}{m_1 m_3^2}.$$

Let $\sigma \in \left(0, \min\left\{\frac{1}{2}, \frac{\theta(1-\delta)}{2m_1 m_3^2}\right\}\right)$, then we get the line search (8). Thus we conclude the result of this lemma. The proof is complete

Lemma 3.4 shows that the line search technique (8) is reasonable, then Algorithm 2.1 is well defined.

Theorem 3.1 Let $\{x_k\}$ be generated by Algorithm 2.1, and let (H1)-(H4) hold. Then, we have

$$\lim_{k\to\infty}\|F_k\| = 0. \quad (33)$$

Proof. According to (8), (20) and Lemma 3.2, we get

$$\begin{aligned} f(x_k+\alpha_k d_k)-R_k &\le \alpha_k^2 \sigma F(x_k)^T d_k \\ &\le -\alpha_k^2 \sigma m_1 \|d_k\|^2 \end{aligned} \quad (34)$$

This means $\lim_{k\to\infty}\alpha_k^2\|d_k\|^2 = 0$.

which implies that

$$\lim_{k\to\infty}\alpha_k = 0. \quad (35)$$

or

$$\lim_{k\to\infty}\|d_k\| = 0. \quad (36)$$

If equation (36) holds, from (7), we get

$$(1-\beta_k)\|F_k\| \le \|F_k\| - \|\gamma_k\| \le \|F_{l(k)}\| - \|\gamma_k\| \le \|B_k d_k\| \le \|B_k\|\|d_k\|.$$

Since B_k is bounded and $\beta_k < 1$, then $\lim_{k\to\infty}\|F_k\| = 0.$

From (8) and Lemma 2.2, for all large enough k, we have

$$\begin{aligned} & f\left(x_k+\frac{\alpha_k}{r}d_k\right)-f_k \\ & > f\left(x_k+\frac{\alpha_k}{r}d_k\right)-R_k > \frac{\alpha_k^2}{r^2}\sigma F(x_k)^T d_k. \end{aligned} \quad (37)$$

Since

$$\begin{aligned} f\left(x_k+\frac{\alpha_k}{r}d_k\right)-f_k &= \frac{\alpha_k}{r}\nabla f(x_k)^T d_k + o\left(\frac{\alpha_k}{r}\|d_k\|\right) \\ &= \frac{\alpha_k}{r}F_k^T \nabla F_k d_k + o\left(\frac{\alpha_k}{r}\|d_k\|\right). \end{aligned} \quad (38)$$

Using this together with (37) and (31), we have

$$\left(\frac{2\sigma}{\theta}-\sigma\frac{\alpha_k}{r}\right)\frac{\alpha_k}{r}F_k^T d_k + o\left(\frac{\alpha_k}{r}\|d_k\|\right) \ge 0. \quad (39)$$

Dividing (39) by $\frac{\alpha_k}{r}\|d_k\|$ and nothing that $\frac{2\sigma}{\theta}-\sigma\frac{\alpha_k}{r} > 0$ and $F_k^T d_k < 0$, we have

$$\lim_{k\to\infty}\frac{F_k^T d_k}{\|d_k\|} = 0. \quad (40)$$

By (20), we have

$$\lim_{k\to\infty}\|d_k\| = 0 \quad (41)$$

So $\lim_{k\to\infty}\|F_k\| = 0$ and the proof is complete.

4. Conclusions

An improved Inexact Newton method is proposed for solving the nonlinear equations in this paper, which use a new nonmonotone inexact Newton method and a new nonmonotone backtracking strategy. Under mild conditions, we obtain the global convergence. But there are at least three issues that need further improvement: (i) The first issue which should be considered is the numerical experiments, the numerical results can demonstrate the efficiency of the new method. (ii) The second issue is the choice of the initial point. It is well known that the initial point plays an important role in an algorithm. (iii) The last important issue is that the proofs of the local convergence need to be completed. All these topics will be the focus of future work.

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