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Efficient Design of IIR Filters with A New Positive Realness based Stability Constraint

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Abstract: A new positive-realness based stability constraint for IIR filters is proposed. Compared to the well-known PR-based condition: { $\text{Re}(D(e^{jw})) \ge e$, for $-p \le w \le p$, where e is a small positive value}, the proposed constraint offers more good choices for the design of stable IIR filters, since only one case of the denominator is excluded where the denominator is in linear phase and the phase spectrum is $\frac{3}{4}p$ or $-\frac{1}{4}p$. With the stability constraint, the design of IIR filters is formulated into a minimax optimization problem which can be solved by a CVX-based procedure. Filters so obtained are guaranteed to be stable and have an approximately equal ripple stopband magnitude response. Examples and comparisons are given to show the effectiveness of the proposed methods.

Keywords: IIR filters; Minimax; Convex optimization; Stability constraint.

1. Introduction

Digital filters have important applications in signal processing. Generally speaking, there are two different classes of digital filter. One is the finite impulse response (FIR) filter where the output signal is finite when the input signal is a unit impulse. The transfer function is a polynomial of the complex variable z . The other is the infinite impulse response (IIR) filter whose transfer function is a rational function. FIR filters are always stable because the impulse response is absolutely summable. For IIR filters, stability is not guaranteed unless all poles of the transfer function lie within the unit circle. FIR filters can be designed with a linear phase response, which cannot be achieved by IIR filters. However, the design of IIR filters generates more attenuation offering the potential of higher stopband attenuation and a lower system delay than their FIR counterparts [1]–[7].

Recently, the exciting possibilities of convex programming have attracted researchers to solving filter design problems using various kinds of convex optimization. It has been shown that the design of FIR filters is easily transformed into a convex form [8,9]. In the case of IIR filters, this is more complex because the transfer function is rational, and it is difficult to write the stability of the filter into a convex constraint. To ensure the stability of filters, different sufficient, or necessary and sufficient conditions for stability have been studied. The best known sufficient condition is a positive-realness based (PR-based) stability constraint, which was first reported in [10] and used in many subsequent works [11]–[18]. A stability constraint based on Rouché's theorem was reported in [19] and [20] which is also a sufficient condition. Some good results may be excluded by making use of these sufficient conditions and IIR filters designed by model-reduction technology are automatically stable [21]. However, the obtained IIR filter is very dependent on the FIR prototype filter and the cutoff frequencies cannot be controlled accurately. In [22] and [23], argument-principle based (AP-based) stability constraints, which are necessary and sufficient conditions, are discussed. A linear equality constraint is achieved by truncating the higher order expansion components. Recently, IIR filters have been designed and the denominator polynomials comprise a cascade of second-order sections [24]–[29]. The necessary and sufficient stability constraints are expressed as a set of linear constraints. However, since the constraints are imposed on the coefficients of these second-order sections, they are difficult to express in convex form with respect to the denominator coefficients. In this work, a new PR-based stability constraint is proposed which definitely offers more good choices for the design of stable IIR filters than the well-known one used in [10]–[18]. Although it is still not a necessary condition, only one case of the denominator is excluded where the denominator is linear phase and the phase spectrum has a constant value ($\frac{3}{4}p$ or $-\frac{1}{4}p$). More, our experiences show no good results can be found if the phase spectrum of linear phase denominators is $\frac{3}{4}p$ or $-\frac{1}{4}p$.

With the proposed stability constraint, the design of causal and stable IIR filters is formulated into a new minimax problem which is solved by CVX, an Matlab-based package for specifying and solving convex programs [30]. In some other minimax designs [13,18,26], sequential procedures are employed to obtain approximately equal ripple filters where the denominator obtained in the k-1th iteration will be used in the kth iteration. In this paper, we employ a constraint on the denominator directly to achieve an approximately equal ripple filter, which is different from the criteria used in [13,18,26]. Similar ideas can be found in [15,20]. However, some hyperbolic inequality constraints are imposed on the autocorrelation coefficients of the denominator there. Examples show better results can be achieved by employing the proposed minimax design criterion and the new PR-based stability constraint.

The paper is organized as follows: The design problem of IIR digital filter is discussed in Section II and the proposed new stability constraint is also studied in this part. In Section III, a design procedure and some examples are given. Conclusions are provided in Section VI.

2. Problem Formulation

2.1. Design of IIR filters in the minimax criteria

The transfer function of causal IIR filters is a rational function which can be written as:

$$H(z)=\frac{\sum_{n=0}^{L_{num}-1} b(n)z^{-n}}{\sum_{n=0}^{L_{den}-1} d(n)z^{-n}} \qquad d(0)=1 \tag{1}$$

where, $b(n)$, $0\le n\le L_{num}-1$, and $d(n)$, $0\le n\le L_{den}-1$, are the coefficients of the numerator and the denominator of the filter. L_{num} and L_{den} denote the lengths of the numerator and the denominator, respectively. The filter order is often defined as $L_{num}-1$. Generally, $L_{den}\le L_{num}$.

By making $z=e^{jw}$, one can get the discrete time Fourier transform (DTFT) of (1):

$$\begin{aligned}H(e^{jw})&=\frac{\sum_{n=0}^{L_{num}-1} b(n)e^{-jwn}}{\sum_{n=0}^{L_{den}-1} d(n)e^{-jwn}}=\frac{B(e^{jw})}{D(e^{jw})}\\ &=\frac{B(w)e^{-jwt_B}}{D(w)e^{-jwt_D}}=H(w)e^{-jwt}\end{aligned} \tag{2}$$

Here, $B(w)e^{-jwt_B}$ is the frequency response of the numerator while $D(w)e^{-jwt_D}$ is the frequency response of the denominator. $H(w)$ and t are the magnitude response and group delay of the filter.

Assume that the frequency response of the desired IIR lowpass filter is:

$$H_d(e^{jw})=\begin{cases}e^{-jwt_d} & w\in[0,w_p]\\ 0 & w\in[w_s,p]\end{cases}, \tag{3}$$

where t_d is the group delay over the passband. The desired response in $w\in[-p,0]$ is $H_d(e^{-jw})$. The design problem considered in this work is to find a causal and stable $H(z)$ that approximates the desired filter $H_d(e^{jw})$. By using the minimax design criterion, the design problem is:

Minimize d_p

$$\begin{aligned}&\text{s.t.}\ |H(e^{jw})-H_d(e^{jw})|\le d_p \quad w\in[0,w_p]\\ &|H(e^{jw})|\le d_s \quad w\in[w_s,p]\end{aligned} \tag{4}$$

where d_p and d_s denote, respectively, the allowed ripples in the considered frequency regions. Since the filter $H(z)$ is a rational function, $H(e^{jw})$ in (4) is now substituted by $\frac{B(e^{jw})}{D(e^{jw})}$:

Minimize d_p

$$\begin{aligned}&\text{s.t.}\ \left|\frac{B(e^{jw})}{D(e^{jw})}-H_d(e^{jw})\right|\le d_p \quad w\in[0,w_p]\\ &\left|\frac{B(e^{jw})}{D(e^{jw})}\right|\le d_s \quad w\in[w_s,p]\end{aligned} \tag{5}$$

It is not convex. The frequency response of the filter is desired to be: $\frac{B(e^{jw})}{D(e^{jw})}=H_d(e^{jw})$. Thus, one can get: $B(e^{jw})=H_d(e^{jw})D(e^{jw})$. Precisely,

$$B(e^{jw})=\begin{cases}=e^{-jwt_d}D(e^{jw}) & w\in[0,w_p]\\ 0 & w\in[w_s,p]\end{cases}, \tag{6}$$

Therefore, the problem in (5) can be relaxed to be a convex form:

$$\begin{aligned}&\text{s.t.}\ |B(e^{jw})-H_d(e^{jw})D(e^{jw})|\le d, w\in[0,w_p]\\ &|B(e^{jw})|\le d_s \quad w\in[w_s,p]\end{aligned} \tag{7}$$

However, there are no constraints applied to $D(e^{jw})$ in the stopband. The filter obtained by solving (7) is generally not an equal ripple one. In [13,18,26], sequential procedures are employed and the denominator obtained in the k-1th iteration will be used in the kth iteration to obtain an approximately equal ripple magnitude response. In this work, a constraint is applied directly to $|D(e^{jw})|$ on the stopband to achieve a filter with an approximately equal ripple stopband magnitude. To do this, the following constraint is used: $|D(e^{jw})|\le d_d$, $w\in[w_s,p]$, where d_d is a given positive value. Experimental results show that filters cannot achieve approximately equal ripple stopband attenuations with a large value of d_d because the constraint on the denominator is too loose. Otherwise, if d_d is given a small value, it is difficult to obtain filters with a good magnitude response because the constraint on the denominator is too tight. The suggested value of d_d is between 0.5 and 5.

Note, the filter obtained by solving the problem in (7) may not be stable because there is no stability condition involved. In the next part, a new PR-based stability condition is studied and incorporated into the design problem to obtain stable IIR filters.

2.2. New positive realness-based stability condition

An IIR filter is stable if there is no pole outside or on the unit circle:

$$D(z) \neq 0 \quad z \geq 1 \tag{8}$$

Thus, the filter is stable if and only if the polynomial $D(z^{-1})$ has no zero for $z \leq 1$. Let $z = re^{jw}$, where $j = \sqrt{-1}$. One gets

$$D(re^{jw}) = \sum_{n=0}^{L_{den}-1} d(n) r^n e^{jwn} \neq 0 \tag{9}$$

for $0 \leq r \leq 1 \quad -p \leq w \leq p$

It is well known that for a complex number to equal zero, its real and imaginary parts cannot both be zero at the same time. In (9), $D(re^{jw})$ cannot be zero means that the real and imaginary parts cannot be zero at the same time. Respectively, denote the real and imaginary parts of $D(re^{jw})$ as:

$$\begin{aligned} \mathrm{Re}[D(re^{jw})] &= \sum_{n=0}^{L_{den}-1} d(n) r^n \cos(wn) \\ \mathrm{Im}[D(re^{jw})] &= \sum_{n=0}^{L_{den}-1} d(n) r^n \sin(wn) \end{aligned} \tag{10}$$

for $0 \leq r \leq 1 \quad -p \leq w \leq p$

There are two possibilities for the relationship between the real and imaginary parts of $D(re^{jw})$.

Case 1:

$$\mathrm{Re}[D(re^{jw})] = -\mathrm{Im}[D(re^{jw})] \neq 0 \tag{11}$$

In this case, the denominator of desired filter $H_d(e^{jw})$ is $D(e^{jw}) = \pm(1-j)\cdot \left| \sum_{n=0}^{L_{den}-1} d(n)\cos(wn) \right|$. It is linear phase and the phase spectrum is constant: $\frac{3}{4}p$ or $-\frac{1}{4}p$. Experiences demonstrate that no elegant results can be found due to the stringent condition for the denominator.

Case 2:

$$\mathrm{Re}[D(re^{jw})] \neq -\mathrm{Im}[D(re^{jw})] \tag{12}$$

In this case, the condition in (9) can now be written as:

$$\begin{aligned} &\mathrm{Re}[D(re^{jw})] + \mathrm{Im}[D(re^{jw})] \\ &= \sum_{n=0}^{L_{den}-1} d(n) r^n (\cos(wn) + \sin(wn)) \\ &= d_0 + \sum_{n=1}^{L_{den}-1} d(n) r^n (\cos(wn) + \sin(wn)) \neq 0 \end{aligned} \tag{13}$$

for $0 \leq r \leq 1 \quad -p \leq w \leq p$

That is, $\mathrm{Re}[D(re^{jw})] + \mathrm{Im}[D(re^{jw})]$ have no zero crossing point in the closed region { $0 \leq r \leq 1$, $-p \leq w \leq p$ }. It must be always either positive or negative. Generally, the first denominator coefficient is set to be $d_0 = 1$. On the original point $r = 0$, $\mathrm{Re}[D(re^{jw})] + \mathrm{Im}[D(re^{jw})] = d_0 = 1$. So,

$$\begin{aligned} &\mathrm{Re}[D(re^{jw})] + \mathrm{Im}[D(re^{jw})] \\ &= d_0 + \sum_{n=1}^{L_{den}-1} d(n) r^n (\cos(wn) + \sin(wn)) > 0 \end{aligned} \tag{14}$$

for $0 \leq r \leq 1 \quad -p \leq w \leq p$

A convex constraint is easily achieved:

$$-\sum_{n=1}^{L_{den}-1} d(n) r^n (\cos(wn) + \sin(wn)) < 1 \tag{15}$$

for $0 \leq r \leq 1 \quad -p \leq w \leq p$

Or, it also can be denoted as:

$$-\sum_{n=1}^{L_{den}-1} d(n) r^n (\cos(wn) + \sin(wn)) \leq 1 - e \tag{16}$$

for $0 \leq r \leq 1 \quad -p \leq w \leq p$

where e is a small positive number.

Compared to the well-known PR-based condition: { $\mathrm{Re}(D(e^{jw})) \geq e$, for $-p \leq w \leq p$, where e is also a small positive value}, it is clear that our proposed new condition almost encompasses that one, except for the case of { $\mathrm{Re}[D(e^{jw})] = -\mathrm{Im}[D(e^{jw})] \geq e$, for $-p \leq w \leq p$ }, where the phase spectrum of the denominator is $-\frac{1}{4}p$. For the new PR-based condition, $\mathrm{Re}(D(e^{jw}))$ can be zero, negative or positive, for $-p \leq w \leq p$, only if $\mathrm{Re}[D(e^{jw})] \neq -\mathrm{Im}[D(e^{jw})]$. Therefore, the proposed new condition naturally offers more good choices for designing stable IIR filters. It is still not a necessary condition, however, but only one kind of filter is excluded, where the phase spectrum of the denominator equals $\frac{3}{4}p$ or $-\frac{1}{4}p$. Also, our experiences have shown no good results can be found if the phase spectrum of the denominator equals $\frac{3}{4}p$ or $-\frac{1}{4}p$. Thus, the proposed new positive realness-based stability condition can be seen as an almost necessary and sufficient one.

By incorporating the new stability constraint into the design problem (7), stable IIR filters will be obtained:

$$\text{Minimize } d \tag{17}$$

$$\text{s.t.} \left| B(e^{jw}) - H_d(e^{jw}) D(e^{jw}) \right| \leq d \quad w \in [0, w_p] \tag{17a}$$

$$\left| B(e^{jw}) \right| \leq d_s \quad \left| D(e^{jw}) \right| \leq d_d \quad w \in [w_s, p] \tag{17b}$$

$$-\sum_{n=1}^{L_{den}-1} d(n) r^n (\cos(wn) + \sin(wn)) < 1 \tag{17c}$$

$$0 \leq r \leq 1 \quad -p \leq w \leq p$$

2.3. Discussion on the setting of r

To solve problem (17) by CVX, $r \in [0,1]$ and $w \in [0,p]$ should be sampled into discrete grid points: $\boldsymbol{r} = \{r_i \mid i = 1,2,...,N_r\}$ and $\boldsymbol{\omega} = \{w_i \mid i = 1,2,...,N_w\}$, respectively. In this paper, N_w is set as: $N_w = 500$, that is: the angular frequency w is sampled uniformly into 500 points in the range $[0,p]$. Now, let us discuss the setting of $\boldsymbol{r}$: $\boldsymbol{r} = \{r_i \mid i = 1,2,...,N_r\}$, for $0 \le r \le 1$. Theoretically speaking, the value of N_r should be large enough to ensure the stability of the filter inside the closed region: $\{ 0 \le r \le 1,\ -p \le w \le p \}$. Abundant design experiments show that N_r does not need to be a large integer. $N_r = 2$ or 3 is ok in most cases. If $N_r = 3$, then $\boldsymbol{r} = \{0, 0.5, 1\}$. The stability condition in (17c) is automatically satisfied when $r = 0$. Therefore, the constraints in equation (17c) are effective when $r = 0.5$ and $r = 1$. If there are only two elements in the setting of $\boldsymbol{r}$: $\boldsymbol{r} = \{0,1\}$, the stability condition in (17c) is only applied for the unit circle: $|z| = 1$. Some works show an undesired bump will be created on the transition band [17,27,29]. Fortunately, the bump can be eliminated by making use of a smaller allowed maximum pole radius [14,15,16,28]. In this work, the outer boundary of $\boldsymbol{r}$ can be relaxed to a larger value, such as 1.05, to offer a smaller allowed maximum pole radius ($\frac{1}{1.05} \approx 0.95238$). Thus, bumps possibly emerged on the transition band of IIR filters are easily eliminated by the proposed stability constraint.

In the next section, a design procedure and some examples are given to show the effectiveness of the proposed method.

3. Design Procedure and Examples

Generally speaking, there is a tradeoff among the filter length, the transition bandwidth, the ripple ratios between the passband and the stopband. The desired passband group delay and the allowed maximum pole radius also produce an effect on the designed filter. To achieve good results, and to improve the real-time performance of systems, the desired passband group delays of examples in this work are all not chosen the case which is larger than the length of the numerator.

To solve the problem in (17), a CVX based design procedure is given here. With the given filter length and cutoff frequencies, the ripple ratios between the passband and the stopband are optimized iteratively to improve the magnitude performance of the designed filters. Denote $\{ E_p, E_{pp}, E_s \}$ as the passband maximum ripple error, the passband peak-to-peak ripple error and the stopband maximum ripple error of designed filter, respectively:

$E_p = \max |H(e^{jw}) - e^{-jwt_d}|$, for passband;

$E_{pp} = \max |H(e^{jw})| - \min |H(e^{jw})|$, for passband;

$E_s = \max |H(e^{jw})|$, for stopband.

Design Procedure:

Given the length of the numerator and denominator of the desired filter, $\{ L_{num}, L_{den} \}$, the passband group delay t_d, the passband and stopband cutoff frequencies $\{ w_p, w_s \}$. Initialize the desired stopband attenuation ripple error d_s and the step sizes to update d_s as $d_s = kd_s$. The suggested value of d_d is between 0.5 and 5 for most cases. The flag to stop the iteration is initialized as $t = 0$.

Step 1.Solve the filter $H(z)$ in (17) using CVX with the given d_s.

Step2.Calculate $\{ E_p, E_{pp} \}$. If the value of E_p or E_{pp} meets the requirement, $\{t = 1$, stop$\}$, Else {update d_s as $d_s = kd_s$ and go back to step 1.}

The stopband attenuation d_s is initialized to a small value, such as: $1e-5$, and k is set to be $k = 5$ to update d_s. Of course, k could also be set to another small constant value (greater than 1) to carry out more a more meticulous search. If the maximum passband ripple of the designed filter $H(z)$ should be smaller than $1e-3$, the condition to stop the search is set as: {if $E_p < 1e-3$, $t = 1$}. If we want to obtain a filter with almost the same level of maximum ripples on the passband and the stopband, the condition to stop the search can be set as: {if $E_p < E_s$, $t = 1$}.

Example 1: Design of IIR highpass filters.

Recently, some good IIR filters have been obtained by various effective methods. In Table I, highpass filters F-1[15], F-1[17] and F-1[23] are taken from Refs. [15] (Example 2), [17] (Example 2) and [23] (Example2), respectively. All of them are designed with the same numerator and denominator lengths
($L_{num} = L_{den} = 15$), the same desired passband group delay ($t_d = 12$) and the same passband and stopband cutoff frequencies ($w_p = 0.525p$, $w_s = 0.475p$). The ideal frequency responses are all defined as

$$H_d(e^{jw}) = \begin{cases} 0 & w \in [0, 0.475p] \\ e^{-jw12}, & w \in [0.525p, p] \end{cases}.$$

For the purpose of comparison, the problem in (17) is solved to design the filter with the same specifications as filters F-1[15], F-1[17] and F-1[23]. To start the proposed design procedure, $\boldsymbol{r}$ is set as: $\boldsymbol{r} = \{0,1\}$ and the condition to stop the design procedure is "if $E_p < E_s$". For this example, $d_d = 3$ is employed which is the constraint applied to $|D(e^{jw})|$ on the stopband to achieve an approximately equal ripple stopband magnitude. The obtained filter is listed in Table I (see filter F-1). Except E_p, E_s and E_{pp}, the bump on the transition band E_t ($=\max(20\log_{10}(|H(e^{jw})|))$, $w \in [0.475p, 0.525p]$) is also listed in Table I to provide more details of the designed filters. The numerator and the denominator coefficients of F-1 are listed in Table II.

Table 1. Highpass IIR filters in example 1.

Filters	L_{num}	L_{den}	t_d	E_p	E_{pp}	E_s	E_t **(dB)**
F-1	15	15	12	3.385e-2	6.737e-2	3.678e-2	N/A
F-1[15]	15	15	12	4.236e-2	N/A	4.285e-2	N/A
F-1[17]	15	15	12	1.552e-2	N/A	1.552e-2	5.90 dB
F-1[23]	15	15	12	4.088e-2	N/A	7.027e-2	N/A

*N/A: not applicable

Table 2. Coefficients of IIR filter F-1.

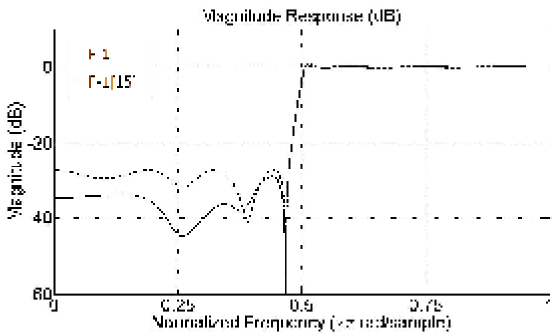
Numerator	**Denominator**
-0.00967936265271	1.00000000000000
-0.00967936265271	0.89982779646765
0.00691948898891	1.11997450935959
-0.00463267633413	0.41036938182165
-0.00978577967759	-0.16974361323017
0.00695845174001	-0.31973744613569
0.01460139359578	-0.11032907590314
-0.01347717317742	0.09982893958563
-0.02442787382744	0.11020444305924
0.03516744736465	0.01064336480125
0.05153034136701	-0.04285466854822
-0.19895112933231	-0.02356938118848
0.28117009077321	0.00515869030614
-0.22735146744700	0.00870968191211
0.11085257713492	0.00151737468286

It is shown that the proposed filter F-1 offers a magnitude response with slightly smaller maximum ripples on both the passband and the stopband, 8.51e-3 and 6.07e-3, respectively, than filter F-1[15]. The magnitude responses and the passband ripples of filters F-1 (solid line) and F-1[15] (dashed-line) are plotted in Figs. 1(a) and 1(b), while the passband group delays are illustrated in Fig. 1(c).

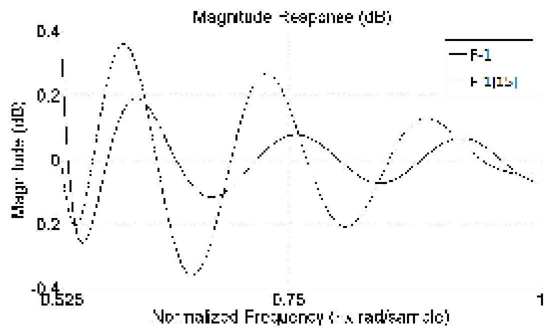
Compared to filter F-1[17], our proposed F-1 has slightly larger ripples in the passband and the
stopband. However, there is a large bump (almost 5.90dB) on the transition band for F-1[17] which is denoted as E_t in Table I. It is clear in Fig. 1(a) there is no such bump on filter F-1.

In [15] and [17], the best known sufficient PR-based stability constraint is employed. The proposed filter F-1 is designed by making use of the new PR-based stability constraint. It is stable, however it does not satisfy the condition $\{\mathrm{Re}(D(e^{jw})) > 0$, for $0 \le w \le p\}$.

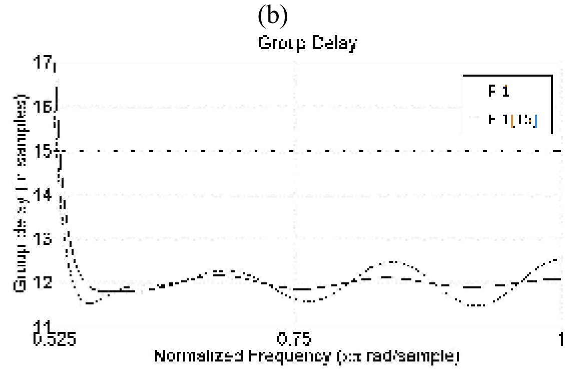
Filter F-1[23] is designed by employing the necessary and sufficient AP-based stability constraint. However, our proposed F-1 has a slightly smaller
passband maximum ripple and a higher stopband attenuation by 7.03e-3 and 3.349e-2, respectively. Although the proposed new PR-based stability constraint is not a necessary one, only one case of the denominator (with a phase spectrum as $\frac{3}{4}p$ or $-\frac{1}{4}p$) is excluded. Thus, good results can be founded by making use of the new PR-based condition.



(a)



(b)



(c)

Figure 1. (a) Magnitude responses, (b) passband ripples and (c) passband group delays of filters F-1 (solid line) and F-1[15] (dashed-line) in example 1

Example 2: Design of IIR highpass filters.

For more comparisons, some other highpass IIR filters were designed by the procedure and summarized as filters F-2, F-3 and F-4 in Table III. The passband and the stopband cutoff frequencies of the three filters are still set as $(w_p, w_s) = (0.525p, 0.475p)$. The different numerator and denominator lengths (L_{num}, L_{den}) and group delays t_d are given in Table III. $\boldsymbol{r}$ is still set as: $\boldsymbol{r} = \{0,1\}$. The design procedure is stopped if $E_{pp} > 1e-1$ for all filters in this example.
In Table III, filters F-2[28], F-3[28] and F-4[28] are all taken from Ref. [28] (Example 1). The denominators of all filters comprise a cascade of second-order sections and a set of linear constraints are imposed on the coefficients of the second-order sections to ensure the stability of designed filters. It seems clear that our proposed stable IIR filters all have a flatter passband and a higher stopband attenuation than their counterparts designed by the method in [28].

Table 3. Highpass IIR filters in example 2.

Filters	L_{num}	L_{den}	t_d	E_p	E_{pp}	E_s
F-2	15	9	11	5.414e-2	9.538e-2	5.573e-2
F-3	15	9	12	4.333e-2	6.530e-2	6.634e-2
F-4	15	10	12	4.004e-2	7.817e-2	5.577e-2
F-2[28]	15	9	11	1.141e-1	N/A	1.492e-1
F-3[28]	15	9	12	1.051e-1	N/A	1.446e-1
F-4[28]	15	10	12	1.142e-1	N/A	1.459e-1

*N/A: not applicable

Example 3: Design of IIR lowpass filters.

In this example, several lowpass IIR filters are designed and denoted as filters F-5, F-6, F-7 and F-8 in Table IV. Maximum allowed pole radii are {0.95, 1, 1 and 0.95} in the design of filters F-5, F-6, F-7 and F-8 respectively. As a comparison, filters designed with the same numerator and denominator lengths and the same passband and stopband edge frequencies as in Refs. [15], [17], [20], [23] and [29] are also summarized as F-5[17] (Example 3 in [17]), F-6[20] (Example 1 in [20]), F-7[15] (Example 1 in [15]), F-7[23] (Example 3 in [23]), F-8[29] (Example 4 in [29]) in Table IV. In the design of stable filters, the best known PR-based stability constraint, the Rouché's theorem-based one, the AP-based one and a set of linear constraint which is imposed on the cascaded second-order denominator factors are respectively used in Refs. [15], [17], [20], [23] and [29].
It can be seen that filters F-5, F-6 and F-7 designed by employing the new PR-based stability constraint have a better magnitude response than their counterparts taken from Refs. [15], [17], [20] and [23], respectively. More details can be found in Table IV. For the page-limitation, only the coefficients of the proposed F-7 are listed in Table V.
For the case of filter F-8, F-8[29] has a comparable passband flatness and a higher stopband attenuation by 1.723 dB. However, filter F-8[29] has a larger bump on the transition band (4.37dB vs. 0.75dB for our proposed filter F-8). The magnitude response of filter F-8 is illustrated in Fig.2. It is easy to see that filter F-8 has an approximate equal ripple stopband magnitude and there is no large bump on the transition band.

Table 4. Lowpass IIR filters in example 3.

Filters	w_p	w_s	L_{num}	L_{den}	t_d	E_p	E_{pp}	E_s
F-5	0.4	0.6	33	17	16	3.438e-5	6.279e-5	2.004e-4
F-5[17]	0.4	0.6	33	17	16	2.30e-4	N/A	2.30e-4
F-6	0.5	0.55	19	19	15	2.055e-2	4.104e-2	2.108e-2
F-6[20]	0.5	0.55	19	19	15	2.121e-2	N/A	2.188e-2
F-7	0.4	0.56	16	5	12	5.051e-3	9.660e-3	5.101e-3
F-7[15]	0.4	0.56	16	5	12	5.176e-3	N/A	5.176e-3
F-7[23]	0.4	0.56	16	5	12	2.221e-2	N/A	1.275e-2
F-8	0.5	0.6	13	13	8	1.809e-2	3.036e-2	1.901e-2
F-8[29]	0.5	0.6	13	13	8	N/A	3.098e-2	1.559e-2

*N/A: not applicable

Table 5. Coefficients of IIR filter F-7.

Numerator	Denominator
-0.00221945000172	1.00000000000000
-0.00157947750501	-0.47734556339444
0.00435729720652	0.87972301916863
0.00344920438386	-0.25083760594261
-0.00758097828722	0.07554281229689
-0.00733600237477	
0.01177662936342	
0.01705589122016	
-0.01891086056952	
-0.04548457725767	
0.03684743611165	

0.22233192424049	
0.37065856892558	
0.35923639369678	
0.21060877272977	
0.07043933265817	

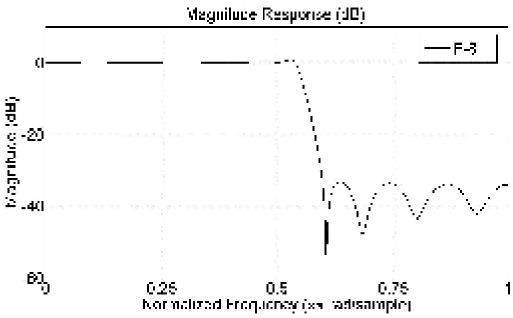


Figure 2. The magnitude response of the proposed filter F-8 in example 3.

Example 4: Design of IIR differentiator.

In the last example, a differentiator is designed with an ideal frequency response given by:

$$H_d(e^{jw}) = \frac{w}{p} e^{j(0.5p-(t_d+0.5)w)}, \quad w \in [0,p]$$

where $t_d = 15$, and the numerator and the denominator lengths are chosen as $L_{num} = L_{den} = 18$, which are the same as the specifications of the differentiator designed in [15] (Example 4). Since the filter is not a fragment constant one, the design problem (17) is simplified as:

$$\text{Minimize } W(w)|B(e^{jw}) - H_d(e^{jw})D(e^{jw})|, \quad w \in [0,p]$$

$$\text{s.t.} \quad -\sum_{n=1}^{L_{den}-1} d(n)r^n(\cos(wn)+\sin(wn)) < 1$$

$$0 \le r \le 1 \qquad -p \le w \le p$$

$W(w)$ is a weight vector to balance the ripples of the magnitude response during the whole frequency region, since there is no ripple ratio between the passband and stopband. In this example $W(w) = p/(p - w + 0.1)$ is employed, where 0.1 is added in case of a division by zero. The allowed maximum radio is 0.97 and the coefficients of obtained differentiator are listed in Table VI.

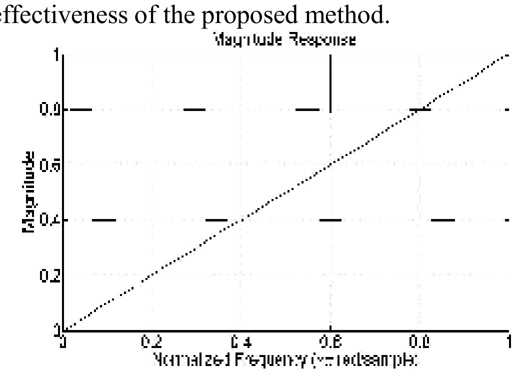
The magnitude response and the group delay are shown in Figs. 3(a) and 3(b). Similar to [15], the group delay has a large error near the origin. However, it is neglected since the frequency almost equals to zero there. The maximum magnitude response ripple of the proposed differentiator is 3.062e-3 and the maximum group delay error within $[0.01p, p]$ is 9.70, while the maximum magnitude response ripple and the group delay error within $[0.01p, p]$ for IIR differentiator in [15] are 3.097e-3 and 9.88, respectively. It seems our proposed method can achieve a comparable result with the method in [15].

Table 6. Coefficients of IIR differentiator in example 4.

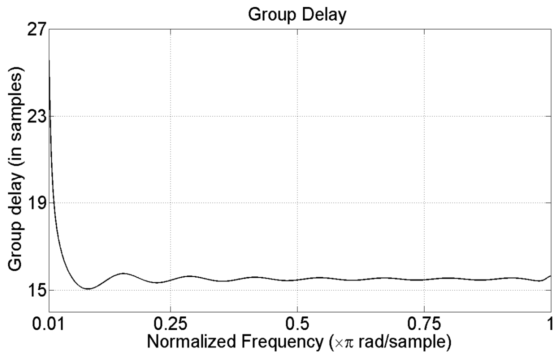
Numerator	Denominator
-0.00263348821027	1.00000000000000
-0.00224598265243	1.02944946769489
-0.00054435678051	0.06381830566475
-0.0000959382657	-0.0101194245961
-0.00023965425544	0.00332237368271
0.00010660097232	-0.00157370904873
-0.00028075593320	0.00069677940145
0.00027557539244	-0.00057137421034
-0.00047246133243	0.00015529752916
0.00061988997763	-0.00034418176725
-0.00104117025774	-0.00004414276325
0.00170990321911	-0.00028655512650
-0.00339224425125	-0.00014862774768
0.00803700314010	-0.00028243216341
-0.02896292438908	-0.00021171440153
0.36004764528265	-0.00028498393955
0.00884142565240	-0.00022090187879
-0.34579770211114	-0.00016505859960

4. Conclusions

A new PR-based stability constraint of IIR filters is developed which naturally offers more good choices for the design of stable IIR filters than the well-known PR-based constraint. By using the proposed stability condition, only one kind of filter is excluded where the phase spectrum of the filter denominator is $\frac{3}{4}p$ or $-\frac{1}{4}p$. The new PR-based stability condition can be seen as an almost sufficient and necessary one, since experiences have shown no good results will be obtained if the phase spectrum of the filter denominator is $\frac{3}{4}p$ or $-\frac{1}{4}p$. A CVX-based design procedure is proposed, and the filters so obtained are stable and have an approximately equal ripple stopband attenuation. Examples are given to show the effectiveness of the proposed method.



(a)



(b)

Figure 3. (a) The magnitude response and (b) the group delay of the proposed IIR differentiator in example 4.

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