One Class Method for Solving One-dimensional Helmholtz Equation with Interface Problem*

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Abstract: Helmholtz equations are often used to characterize the acoustic, electromagnetic scattering, radiation and vibration phenomena building, which is focused on how to solve by many scholars. It will be more difficult to solve Helmholtz equations with discontinuous wave number or singular source term. In this paper, the one-dimensional Helmholtz equation with source term is solved by using the high-order method developed in reference, it can not only keep the local conservation of physical quantity, but also get the desired order. the effectiveness and feasibility of the scheme developed in the paper is demonstrated by numerical examples.

Keywords: Helmholtz equation; Finite volume method; Compact difference scheme; Discontinuous coefficient

1. Introduction

In this paper, we consider the following one-dimensional Helmholtz equation

$$\frac{d^2E}{dz^2}+k_0^2 v(z)E=f(z) \quad or \quad \frac{d^2E}{dz^2}+k^2E=f(z) \quad z\in D \qquad (1)$$

where the material coefficient $v(z)$ is assumed piecewise-constant, where k_0 is a wave number. $D=[Z_{\min},Z_{\max}]$ is a domain.(Among has a jump across interface Γ at the interval) For convenience, we will consider interface on the grid.

The jumps are defined as the difference of the limiting values from two different sides of the interface, for example,

$$[E]\big|_{z=\Gamma}=\lim_{z\to\Gamma\, z\in D^+}E(z)-\lim_{z\to\Gamma\, z\in D^-}E(z)=E^+-E^-. \qquad (2)$$

Early scholars such as Kreiss and Oliger [5] did a lot of work has been on the wave propagation
and also got four or six levels of high precision.Later, Tony [10] proposed a eight order accuracy compact difference scheme. But when the coefficient of only piecewise continuous or interface, to achieve high precision becomes more difficult.For dealing with interface, LeVeque and Zhang [11] for the first time put forward using the virtual point at the interfaces to construct the format.Subsequent papers included[1,2,6,7,8,9].These authors for interface problems are made very good results.G. Baruch, [3] people using finite volume method and finite difference method of one dimensional Helmholtz equation of the source term is zero and the coefficient of discontinuous high-order compact difference scheme is constructed, and they only solve the discontinuity of the grid point.Do this method presented in this paper, further improvement and construct the Helmholtz equation of one dimension with the singular source term in interface (interface on the grid point and the interface fall outside the grid point) of high-order compact format and verify the effectiveness and feasibility of this method.

2. Fourth-order Compact Scheme For The one-dimensional Helmholtz Equation

Let $a,b\in D, a<b.$ We integrate (1) between the points a and b with respect to z:

$$\frac{dE(b)}{dz}-\frac{dE(a)}{dz}+k_0^2\int_a^b v(z)Edz=\int_a^b f(z)dz. \qquad (3)$$

(3) can be interpreted as the integral conservation law that corresponds to (1) for sufficiently smooth solutions, the two formulations are equivalent, see[1].

Without loss of generality, we consider the interval $D=[Z_{\min},Z_{\max}]$. First we generate a mesh: $z_m=Z_{\min}+(m-1)h,\quad where \quad h=\frac{Z_{\max}-Z_{\min}}{M-1},\quad m=1,\mathbf{K},M.$ is the spatial mesh size. In addition, for the material $v(z)$ is a piecewise constant coefficient, so in between each district can be expressed as

$v(z)=v_{m+\frac{1}{2}},\quad z\in(z_m,z_{m+1}).\ v(z)=v_{m-\frac{1}{2}},\quad z\in(z_{m-1},z_m).$

We approximate the Helmholtz equation on a uniform grid with size h by applying the integral relation (3) between the midpoints of every two neighboring cells, i.e., for $[a,b]=[z_{m-\frac{1}{2}}, z_{m+\frac{1}{2}}], m=1,2,\mathbf{K},M$.

Then,

$$\left.\frac{dE}{dz}\right|_{z_{m-\frac{1}{2}}}^{z_{m+\frac{1}{2}}} + k_0^2\int_{z_{m-\frac{1}{2}}}^{z_{m+\frac{1}{2}}} v(z)Edz = \int_{z_{m-\frac{1}{2}}}^{z_{m+\frac{1}{2}}} f(z)dz. \quad (4)$$

Because the material coefficient $v(z)$ has a finite jump across interface z_m on the grid point, so that on the type can be written as:

$$\left.\frac{dE}{dz}\right|_{z_{m-\frac{1}{2}}}^{z_m} + \left.\frac{dE}{dz}\right|_{z_m}^{z_{m+\frac{1}{2}}} + k_0^2 v_{m-\frac{1}{2}}\int_{z_{m-\frac{1}{2}}}^{z_m} Edz + k_0^2 v_{m+\frac{1}{2}}\int_{z_m}^{z_{m+\frac{1}{2}}} Edz$$
$$= \int_{z_{m-\frac{1}{2}}}^{z_m} f(z)dz + \int_{z_m}^{z_{m+\frac{1}{2}}} f(z)dz. \quad (5)$$

The differential equation (1) inside the grid cells can be used to evaluate the one-sided second derivatives at the grid nodes as follows:

$$E''_{m+} \overset{def}{=} \left.\frac{d^2E}{dz^2}\right|_{z=z_{m+}} = -k_0^2 v_{m+\frac{1}{2}} E_{m+} + f_{m+}$$
$$E''_{(m+1)-} \overset{def}{=} \left.\frac{d^2E}{dz^2}\right|_{z=z_{(m+1)-}} = -k_0^2 v_{m+\frac{1}{2}} E_{m+1} + f_{m+1} \quad (6a)$$

$$E''_{m-} \overset{def}{=} \left.\frac{d^2E}{dz^2}\right|_{z=z_{m-}} = -k_0^2 v_{m-\frac{1}{2}} E_{m-} + f_{m-}$$
$$E''_{(m-1)+} \overset{def}{=} \left.\frac{d^2E}{dz^2}\right|_{z=z_{(m-1)+}} = -k_0^2 v_{m-\frac{1}{2}} E_{m-1} + f_{m-1} \quad (6b)$$

We use formulae (6) to approximate each of the four terms on the left-hand side of (5) with fourth-order accuracy.

To approximate the fluxes $E'_{m\pm\frac{1}{2}}$ in (5), we use the Taylor expansion:

$$E'_{m+\frac{1}{2}} = \frac{E_{m+1}-E_m}{h} - \frac{h^2}{24}E^{(3)}_{m+\frac{1}{2}} + O(h^4). \quad (7a)$$

$$E'_{m-\frac{1}{2}} = \frac{E_m-E_{m-1}}{h} - \frac{h^2}{24}E^{(3)}_{m-\frac{1}{2}} + O(h^4). \quad (7b)$$

Respectively weighted average and derivation for (6a) and (6b):

$$E^{(3)}_{m+\frac{1}{2}} = -k_0^2 v_{m+\frac{1}{2}} E'_{m+\frac{1}{2}} + \frac{1}{2}(f'_{m+1}+f'_{m+}),$$
$$E^{(3)}_{m-\frac{1}{2}} = -k_0^2 v_{m-\frac{1}{2}} E'_{m-\frac{1}{2}} + \frac{1}{2}(f'_{m-1}+f'_{m-}). \quad (8)$$

We obtain

$$E'_{m+\frac{1}{2}} = \left(\frac{E_{m+1}-E_m}{h} - \frac{h^2}{48}f'_{m+1} - \frac{h^2}{48}f'_{m+}\right)\Bigg/\left(1-\frac{v_{m+\frac{1}{2}}(hk_0)^2}{24}\right) + O(h^4)$$
$$E'_{m-\frac{1}{2}} = \left(\frac{E_m-E_{m-1}}{h} - \frac{h^2}{48}f'_{m-1} - \frac{h^2}{48}f'_{m-}\right)\Bigg/\left(1-\frac{v_{m-\frac{1}{2}}(hk_0)^2}{24}\right) + O(h^4) \quad (9)$$

Put them together, (5) can be described as

$$\left.\frac{dE}{dz}\right|_{z_{m-\frac{1}{2}}}^{z_m} + \left.\frac{dE}{dz}\right|_{z_m}^{z_{m+\frac{1}{2}}} = \left(\frac{E_{m+1}-E_m}{h} - \frac{h^2}{48}f'_{m+1} - \frac{h^2}{48}f'_{m+}\right)\Bigg/\left(1-\frac{v_{m+\frac{1}{2}}(hk_0)^2}{24}\right)$$
$$-\left(\frac{E_m-E_{m-1}}{h} - \frac{h^2}{48}f'_{m-1} - \frac{h^2}{48}f'_{m-}\right)\Bigg/\left(1-\frac{v_{m-\frac{1}{2}}(hk_0)^2}{24}\right)$$
$$-(E'_{m+}-E'_{m-}) + O(h^4) \quad (10)$$

To approximate the two integral terms in (5), we use cubic interpolating polynomials and approximate the integrand $E(z)$ with fourth-order accuracy.

Lemma 2.1[1] Let $E(z)\in C^6[z_m, z_{m+1}]$. Given its values $\{E_m, E_{m+1}\}$, as well as the values of its one-sided second derivatives $\{E''_{m+}, E''_{(m+1)-}\}$, we can approximate $E(z)$ with fourth-order accuracy:

$$E(z_m + z\,h) = P_3(z) + O(h^4), z\in[0,1] \quad (11a)$$

using the cubic Hermite-Birkhoff polynomial:

$$P_3(z) = (E_m - \frac{h^2}{6}E''_{m+})(1-z) + \frac{h^2}{6}E''_{m+}(1-z)^3 + (E_{m+1} - \frac{h^2}{6}E''_{(m+1)-})z + \frac{h^2}{6}E''_{(m+1)-}z^3 \quad (11b)$$

the polynomial $P_3(z)$ is unique.

Substituted (6a) into (11a), the fourth scheme can be proposed at $[z_m, z_{m+1}]$ for $E(z)$

$$E(z_m+z\,h) = \left[\left(1+\frac{(hk_0)^2}{6}v_{m+\frac{1}{2}}\right)E_{m+} - \frac{h^2}{6}f_{m+}\right](1-z) - \left[\frac{(hk_0)^2}{6}v_{m+\frac{1}{2}}E_{m+} - \frac{h^2}{6}f_{m+}\right](1-z)^3$$
$$+\left[\left(1+\frac{(hk_0)^2}{6}v_{m+\frac{1}{2}}\right)E_{m+1} - \frac{h^2}{6}f_{m+1}\right]z - \left[\frac{(hk_0)^2}{6}v_{m+\frac{1}{2}}E_{m+1} - \frac{h^2}{6}f_{m+1}\right]z^3 + O(h^4) \quad (12)$$

Substituted (12) into the second integral term of (5), we have

$$\int_{z_m}^{z_{m+\frac{1}{2}}} Edz = \frac{3h}{8}\left(1+\frac{(hk_0)^2}{16}v_{m+\frac{1}{2}}\right)E_{m+} + \frac{h}{8}\left(1+\frac{(hk_0)^2}{48}v_{m+\frac{1}{2}}\right)E_{m+1}$$
$$-\frac{3h^3}{128}f_{m+} - \frac{7h^3}{384}f_{m+1} + O(h^5). \quad (13)$$

In the same way, the first integral term in (5) is following:

$$\int_{z_{m-\frac{1}{2}}}^{z_m} Edz = \frac{3h}{8}\left(1+\frac{(hk_0)^2}{16}v_{m-\frac{1}{2}}\right)E_{m-} + \frac{h}{8}\left(1+\frac{(hk_0)^2}{48}v_{m-\frac{1}{2}}\right)E_{m-1}$$
$$-\frac{3h^3}{128}f_{m-} - \frac{7h^3}{384}f_{m-1} + O(h^5). \quad (14)$$

Put all terms of (5) together , the fourth scheme is

$$a1E_{m-1} + c1E_{m-} + c2E_{m+} + a2E_{m+1} - [E'] = g_m. \quad (15)$$

here

$$a1 = \frac{1}{h^2}\Bigg/\left(1-\frac{(hk\_)^2}{24}\right) + \frac{1}{8}k_-^2 + \frac{7}{384}h^2k_-^4 \quad (16)$$

$$c1=-\frac{1}{h^2}\Big/\left(1-\frac{(hk_-)^2}{24}\right)+\frac{3}{8}k_-^2+\frac{3}{128}h^2k_-^4$$

$$c2=-\frac{1}{h^2}\Big/\left(1-\frac{(hk_+)^2}{24}\right)+\frac{3}{8}k_+^2+\frac{3}{128}h^2k_+^4$$

$$a2=\frac{1}{h^2}\Big/\left(1-\frac{(hk_+)^2}{24}\right)+\frac{1}{8}k_+^2+\frac{7}{384}h^2k_+^4$$

$$g_m=\frac{h}{48}(f'_{m+1}+f'_{m+})\Big/\left(1-\frac{(hk_+)^2}{24}\right)-\frac{h}{48}(f'_{m-1}+f'_{m-})\Big/\left(1-\frac{(hk_-)^2}{24}\right)$$
$$+\frac{7(hk_-)^2}{384}f_{m-1}+\frac{3(hk_-)^2}{128}f_{m-}+\frac{3(hk_+)^2}{128}f_{m+}+\frac{7(hk_+)^2}{384}f_{m+1}$$
$$+(F_{m-}-F_{m-\frac{1}{2}})+(F_{m+\frac{1}{2}}-F_{m+}) \tag{17}$$

$$F_{m-}-F_{m-\frac{1}{2}}=\int_{z_{m-\frac{1}{2}}}^{z_m}f(z)dz,F_{m+\frac{1}{2}}-F_{m+}=\int_{z_m}^{z_{m+\frac{1}{2}}}f(z)dz \tag{18}$$

$$k^2=k_0^2v \tag{19}$$

$$[E]=(E_{m+}-E_{m-}),[E']=(E'_{m+}-E'_{m-}) \tag{20}$$

For regular point, the fourth scheme is following

$$aE_{m-1}+bE_m+aE_{m+1}=g1_m. \tag{21}$$

Here

$$a=\frac{1}{h^2}\Big/\left(1-\frac{(hk)^2}{24}\right)+\frac{1}{8}k^2+\frac{7}{384}h^2k^4$$

$$b=-\frac{2}{h^2}\Big/\left(1-\frac{(hk)^2}{24}\right)+\frac{3}{4}k^2+\frac{3}{64}h^2k^4$$

$$g1_m=\frac{h}{48}(f'_{m+1}-f'_{m-1})\Big/\left(1-\frac{(hk)^2}{24}\right)+\frac{(hk)^2}{64}(\frac{7}{6}f_{m-1}+3f_m+\frac{7}{6}f_{m-1})+(F_{m+\frac{1}{2}}-F_{m-\frac{1}{2}}) \tag{22}$$

3. Numerical Experiments

In this section, we present four numerical examples that we have the exact solution to show the convergence of our fourth order compact schemes for solving the Helmholtz equation with a straight interface. All computations are done using a Dell Desktop or a notebook computer. Most of computations are done within seconds or a few minutes depending the mesh size. A Dirichlet boundary condition is used. The error is measured in the $L_\infty=\max_{m=1,\mathbf{K},n+1}|E(z_m)-E_m|$ norm for all the grid points and the convergence order is estimated using $\log(L_\infty h_1/L_\infty h_2)/\log(h_1/h_2)$ as a common practice in the literature.

Example 1 In this example, we use the exact solution $E(z)=\sin(p\,z)$. The wave number has a finite jump across $z=0$, so does $f(z)$ within the domain $D=[-1,1]$ which includes two parts $D^-=[-1,0]$ and $D^+=(0,1]$. The source term is given by

$$f(z)=\begin{cases}(-p^2+k_-^2)\sin(p\,z), & z\in D^-,\\ (-p^2+k_+^2)\sin(p\,z), & z\in D^+.\end{cases} \tag{23}$$

Table 1. A grid refinement analysis for Example 1 using the fourth order compact scheme with different wave numbers. fourth order convergence is confirmed

	$k_-^2=1,k_+^2=5$		$k_-^2=1,k_+^2=30$	
N	Error	Order	Error	Order
8	0.0067		0.0150	
16	4.2991e -004	3.9621	9.5710e -004	3.9701
32	2.7256e -005	3.9794	6.2632e -005	3.9337
64	1.7066e -006	3.9971	3.9343e -006	3.9927
128	1.0671e -007	3.9994	2.4643e -007	3.9969
256	6.6696e -009	4.0000	1.5411e -008	3.9991

In Table 1, we show a grid refinement analysis with different wave numbers. In the second-third column, we show the results with relatively small wave number $k_-^2=1$ and $k_+^2=5$; in the fourth-fifth column, we show the results with medium size wave number $k_-^2=1$ and $k_+^2=30$. In both cases, fourth order convergence can be clearly observed.

Example 2 We consider the following problem: the interface is the point $z=0$ within the domain $D=[-1,1]$ which includes two parts $D^-=[-1,0]$ and $D^+=(0,1]$.

the source item is given as

$$f(z)=\begin{cases}(k_-^2-2)\sin(z)\cos(z-\frac{1}{2})-2\cos(z)\sin(z-\frac{1}{2}), & z\in D^-,\\ (k_+^2-2)\sin(z)\cos(z+\frac{1}{2})-2\cos(z)\sin(z+\frac{1}{2}), & z\in D^+.\end{cases} \tag{24}$$

The exact solution of the problem is

$$E(z)=\begin{cases}\sin(z)\cos(z-\frac{1}{2}), & z\in D^-,\\ \sin(z)\cos(z+\frac{1}{2}), & z\in D^+.\end{cases} \tag{25}$$

Example 3 We consider the following problem: the interface is the point $z=0$ within the domain $D=[-1,1]$ which includes two parts $D^-=[-1,0]$ and $D^+=(0,1]$.

the source item is given as

$$f(z)=\begin{cases}(k_-^2-1)\cos(z)-k_-^2/2, & z\in D^-,\\ (k_+^2-1)\cos(z)-k_+^2/2, & z\in D^+.\end{cases} \tag{26}$$

Table 2. A grid refinement analysis for Example 2 using the fourth order compact scheme with different wave numbers. fourth order convergence is confirmed

	$k_-^2=1, k_+^2=5$		$k_-^2=1, k_+^2=30$	
N	Error	Order	Error	Order
8	8.6281e -004		0.0016	
16	5.6255e -005	3.9390	1.0033e -004	3.9952
32	3.5385e -006	3.9908	6.3711e -006	3.9725
64	2.2181e -007	3.9957	4.0284e -007	3.9878
128	1.3869e -008	3.9994	2.5198e -008	3.9988
256	8.6667e -010	4.0002	1.5762e -009	3.9988

The exact solution of the problem is

$$E(z)=\begin{cases}\cos(z)-\dfrac{1}{2}, & z\in D^-,\\ \cos(z)+\dfrac{1}{2}, & z\in D^+.\end{cases} \qquad (27)$$

The jump condition of the problem is $[E]=1$, $[E']=0$. Unlike Example 2, $E(z)$ is discontinuous at the interface in this example.

Table 3. A grid refinement analysis for Example 3 using the fourth order compact scheme with different wave numbers. fourth order convergence is confirmed

	$k_-^2=1, k_+^2=5$		$k_-^2=1, k_+^2=30$	
N	Error	Order	Error	Order
8	2.0512e -004		7.5973e -005	
16	1.3030e -005	3.9766	5.0548e -006	3.9098
32	8.2297e -007	3.9849	3.2301e -007	3.9680
64	5.1448e -008	3.9985	2.0380e -008	3.9864
128	3.2180e -009	4.0000	1.2757e -009	3.9978
256	1.9702e -010	4.0298	7.9784e -011	3.9990

In Table 3, we show a grid refinement analysis with different wave numbers. In the second-third column, we show the results with relatively small wave number $k_-{}^2=1$ and $k_+{}^2=5$; in the fourth-fifth column, we show the results with medium size wave number $k_-{}^2=1$ and $k_+{}^2=30$. In both cases, fourth order convergence can be clearly confirmed.

Table 4. A grid refinement analysis for Example 4 using the fourth order compact scheme with different wave numbers. fourth order convergence is confirmed.

	$k_-^2=1, k_+^2=5$		$k_-^2=1, k_+^2=30$	
N	Error	Order	Error	Order
8	1.1471e -004		0.0011	
16	7.5972e -006	3.9164	6.5707e-005	4.0653
32	4.8146e -007	3.9800	4.2553e-006	3.9487
64	3.0195e -008	3.9950	2.6626e-007	3.9984
128	1.8863e -009	4.0007	1.6662e-008	3.9982
256	1.0531e -010	4.1628	1.0387e-009	4.0037

Example 4 We consider the following problem, the interface is the point $z=0$ within the domain $D=[-1,1]$ which includes two parts $D^-=[-1,0]$ and $D^+=(0,1]$. the source item is given as

$$f(z)=\begin{cases}(k_-^2-1)z\cos(z)-2\sin(z)-k_-^4(z-\dfrac{1}{2}), & z\in D^-,\\ (k_+^2-1)z\cos(z)-2\sin(z)+k_+^4(z+\dfrac{1}{2}), & z\in D^+.\end{cases} \qquad (28)$$

The exact solution of the problem is

$$E(z)=\begin{cases}z\cos(z)-k_-^2(z-\dfrac{1}{2}), & z\in D^-,\\ z\cos(z)+k_+^2(z+\dfrac{1}{2}), & z\in D^+.\end{cases}$$

The jump condition of the problem is $[E]=\frac{1}{2}(k_+^2-k_-^2), \qquad [E']=k_+^2+k_-^2.$

4. Conclusions

In this paper, the finite volume method in the literature on the basis of existing methods, the method was further improved, the one dimensional Helmholtz equation of wave number discrete fourth order accuracy difference scheme. Numerical experiment results show that the format can be up to four order accuracy.

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References

[1] Feng Xiufang, "A high-order compact scheme for the one-dimensional Helmholtz equation with a discontinuous coefficient", International Journal of Computer Mathematics, vol 89,pp. 618-624,2010.

[2] Feng XF, Li ZL and Qiao ZH, "High order compact finite difference schemes for the Helmholtz equation with discontinuous coefficients", J. Comput. Math,vol 29(3), pp. 324-340,2011.

[3] G. Baruch, G. Fibich,,S. Tsynkov and E.Turkel. "Fourth Order Schemes for Time-Harmonic Equations with Discontinuous Coefficients", Commun. Comput. Phys.,2009.

[4] G. Baruch, G. Fibich and S. Tsynkov, "High-order numerical method for the nonlinear Helmholtz equation with material discontinuities", J. Comput. Phys., 2007. Submitted for publication.

[5] H.-O. Kreiss, J Oliger, "Comparison of accurate methods for the integration of hyperbolic equations", Tellus 24,pp.199–215,1972.

[6] H.-O. Kreiss ,N.A. Peterson, "An embedded boundary method for the wave equation with disc -ontinuous coefficients", UCRL-JRNL 215702, Lawrence Livermore National Lab,2005.

[7] J-R Li ,L. Greengard, "High order marching schemes for the wave equation in complex geometry", J. Comput. Phys, vol 198(1),pp. 295–309,2004.

[8] S. Deng,W. Cai, "A Fourth-Order Upwinding Embedded Boundary Method (UEBM) for Max-

well's Equations in Media with Material Interfaces: Part I", Commun. Comput. Phys, vol 21,pp.1-21,2006.

[9] S. Zhao, G.W. Wei, "High-order FDTD methods via derivative matching for Maxwell's equati-ons with material interfaces", J. Comput. Phys, vol 200,pp.60-103,2004.

[10] Tony W. H. Sheu, C. F. Chen and L. W. Hsieh, "A highly accurate Helmholtz scheme for modeling Scat tering wave", Computer Methods in Applied Mechanics and Engineering, vol 193,pp.5573-5583,2004.

[11] Zhang Chaoming ,R.J. LeVeque, The immersed interface method for acoustic wave equations

with discontinuous coefficients", Wave Motion, vol 25, pp.237-263,1997.